ECEn 670

Homework Problem Set 3

Due at beginning of class, Thursday, October 8, 2009

Problems are from *An Introduction to Statistical Signal Processing* by Gray and Davisson unless otherwise specified.

1. 3.14
2. 3.18
3. 3.20
4. 3.21
5. 3.22
6. 3.23
7. 3.35
8. 3.38
9. 3.39
10. 3.43
11. 3.48
12. 3.55

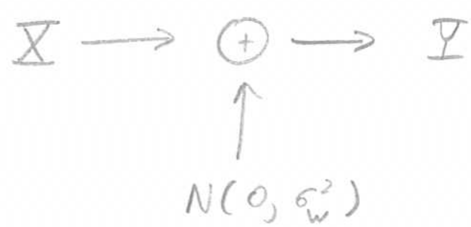
3.14) $X \to \{a, b\}$

$p_X(a) = p$

$p_X(b) = 1-p$

$Y: \quad f_{Y|X}(y|x) = \dfrac{e^{-(y-x)^2/2\sigma_w^2}}{\sqrt{2\pi\sigma_w^2}}$

We see that this is a situation where we have added Gaussian noise



We have $f_{Y|X}(y|x)$.

For our MAP detector we need $f_{X|Y}(x|y)$. Let's use Bayes' Rule.

Using (3.92)

$$p_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)\, p_X(x)}{\sum_\alpha p_X(\alpha) f_{Y|X}(y|\alpha)} = \frac{\left(\dfrac{e^{-(y-a)^2/2\sigma_w^2}}{\sqrt{2\pi\sigma_w^2}}\right) p\,\delta(x-a) + \left(\dfrac{e^{-(y-b)^2/2\sigma_w^2}}{\sqrt{2\pi\sigma_w^2}}\right)(1-p)\,\delta(x-b)}{p\left[\dfrac{e^{-(y-a)^2/2\sigma_w^2}}{\sqrt{2\pi\sigma_w^2}}\right] + (1-p)\left[\dfrac{e^{-(y-b)^2/2\sigma_w^2}}{\sqrt{2\pi\sigma_w^2}}\right]}$$

$$= \frac{p\, e^{-(y-a)^2/2\sigma_w^2}\,\delta(x-a) + (1-p)\, e^{-(y-b)^2/2\sigma_w^2}\,\delta(x-b)}{p\, e^{-(y-a)^2/2\sigma_w^2} + (1-p)\, e^{-(y-b)^2/2\sigma_w^2}}$$

$$p_{X|Y}(a|y) = \frac{p\, e^{-(y-a)^2/2\sigma_w^2}}{p\, e^{-(y-a)^2/2\sigma_w^2} + (1-p)\, e^{-(y-b)^2/2\sigma_w^2}}$$

$$p_{X|Y}(b|y) = \frac{(1-p)\, e^{-(y-b)^2/2\sigma_w^2}}{p\, e^{-(y-a)^2/2\sigma_w^2} + (1-p)\, e^{-(y-b)^2/2\sigma_w^2}}$$

3.14 cont...)

Our MAP detector (3.96) will be

$$\hat{X}(y) = \arg\max_{x} \; p_{X|Y}(x|y)$$

$$= \arg\max_{x} \left[p e^{-(y-a)^2/2\sigma_w^2} \delta(x-a) + (1-p) e^{-(y-b)^2/2\sigma_w^2} \delta(x-b) \right]$$

We see there is a threshold when $p_{X|Y}(a|y) = p_{X|Y}(b|y)$

$$p e^{-(y-a)^2/2\sigma_w^2} = (1-p) e^{-(y-b)^2/2\sigma_w^2}$$

$$\ln p - \frac{(y-a)^2}{2\sigma_w^2} = \ln(1-p) - \frac{(y-b)^2}{2\sigma_w^2}$$

$$2\sigma_w^2 (\ln p - \ln(1-p)) = (y-a)^2 - (y-b)^2$$

$$= \cancel{y^2} - 2ay + a^2 - \cancel{y^2} + 2by - b^2$$

$$= y(2b-2a) + a^2 - b^2$$

$$y_{th} = \frac{2\sigma_w^2 (\ln p - \ln(1-p)) - a^2 + b^2}{2b-2a}$$

This is the threshold where we decide between a and b.

$$P_e = P_r(\hat{X}(Y) \neq X)$$

$$= P_r(\hat{X}(Y) \neq a \mid X = a)\, p_X(a) + P_r(\hat{X}(Y) \neq b \mid X = b)\, p_X(b)$$

let's say $a \leq b$

$$= P_r(Y > y_{th} \mid X = a)\, p_X(a) + P_r(Y < y_{th} \mid X = b)\, p_X(b)$$

$$\left\{ \begin{aligned} &\Phi(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-u^2/2}\, du \\ &Q(\alpha) = 1 - \Phi(\alpha) \end{aligned} \right\}$$

$$= Q\left(\frac{y_{th} - a}{\sigma_w}\right) p_X(a) + Q\left(\frac{b - y_{th}}{\sigma_w}\right) p_X(b)$$

$$P_e = \left[Q\left(\frac{|y_{th} - a|}{\sigma_w}\right)\right] p + \left[Q\left(\frac{|b - y_{th}|}{\sigma_w}\right)\right](1-p)$$

3.14 cont...)

If $p = 0.5$, this makes things much easier. Want to maximize distance between a and b. The threshold will be at $\frac{a+b}{2}$.

If $(a^2+b^2)/2 = E_b$,

and $a = -b$, $\frac{2a^2}{2} = E_b \rightarrow a = \pm\sqrt{E_b}$ $\qquad |a-b| = 2\sqrt{E_b}$

If $b = 0$,

$\frac{a^2}{2} = E_b \Rightarrow a = \sqrt{2E_b}$ $\qquad |a-b| = \sqrt{2E_b}$

Thus, it is a much better choice under these constraints that $a = -b$.

We can see that the P_e will be minimized in this case and the threshold will be at zero. Thus

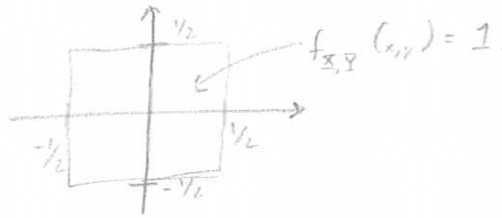
$$P_e = \left[Q\left(\frac{\sqrt{E_b}}{\sigma_w}\right)\right](0.5) + \left[Q\left(\frac{\sqrt{E_b}}{\sigma_w}\right)\right](0.5)$$

$$= Q\left(\frac{\sqrt{E_b}}{\sigma_w}\right)$$

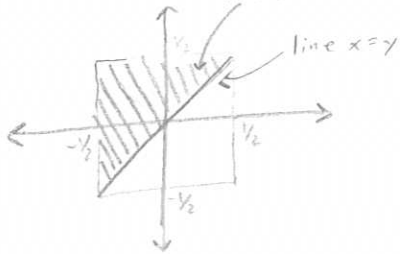
3.18) a)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} c\,dx\,dy = c = 1$$

$\therefore c = 1.$



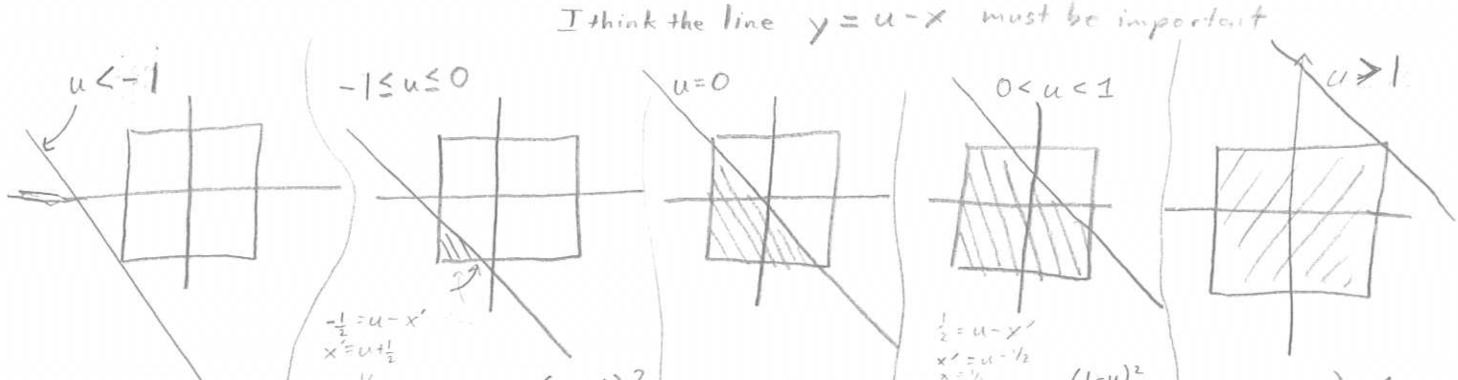
b) $P(\{x,y : x<y\}) = \iint_{Area} f_{X,Y}(x,y) = \frac{1}{2}(1)(1)c = \underline{\frac{1}{2}}$



c) $U(x,y) = x+y$

$F_U(u) = Pr(U \le u) = Pr(x+y \le u)$

I think the line $y = u-x$ must be important



$P(U \le u) = 0$ | $P(U \le u) = \frac{(u+1)^2}{2}$ | $P(U \le u) = \frac{1}{2}$ | $P(U \le u) = 1 - \frac{(1-u)^2}{2}$ | $P(U \le u) = 1$

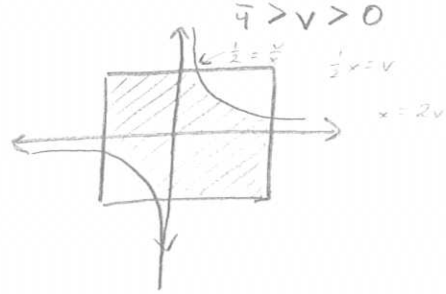
$$P(U \le u) = \begin{cases} 0, & u < -1 \\ \frac{(u+1)^2}{2}, & -1 \le u \le 0 \\ 1 - \frac{(1-u)^2}{2}, & 0 < u \le 1 \\ 1, & 1 < u \end{cases}$$

3.18 cont...)

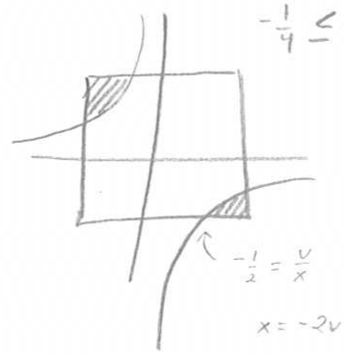
d) $V: \mathbb{R}^2 \to \mathbb{R}$ $\quad V(x,y) = xy$ $\quad$ Find $F_V(v) = \Pr(V \le v) = \Pr(xy \le v)$

$y = \frac{v}{x}$ is the important equation

$\frac{1}{4} > v > 0$



$-\frac{1}{4} \le v < 0$



For $0 < v \le \frac{1}{4}$

$$F_V(v) = \frac{1}{4} + \frac{1}{4} + 2\left[\int_{2v}^{1/2} \frac{v}{x}\,dx + \int_0^{2v} \frac{1}{2}\,dx\right]$$

$$= \frac{1}{2} + 2\left[v + \left[v \ln x\right]_{2v}^{1/2}\right]$$

$$= \frac{1}{2} + 2v + 2v\left[\ln\left(\tfrac{1}{2}\right) - \ln(2v)\right]$$

$$= \frac{1}{2} + 2v - 2v\ln(4v)$$

For $-\frac{1}{4} \le v < 0$

$$F_V(v) = 2\int_{-2v}^{1/2}\left(\frac{v}{x} + \frac{1}{2}\right)dx = 2\left[v\ln x + \tfrac{1}{2}x\right]_{-2v}^{1/2}$$

$$= 2\left[v\ln\left(\tfrac{1}{2}\right) + \tfrac{1}{4} - v\ln(-2v) + v\right]$$

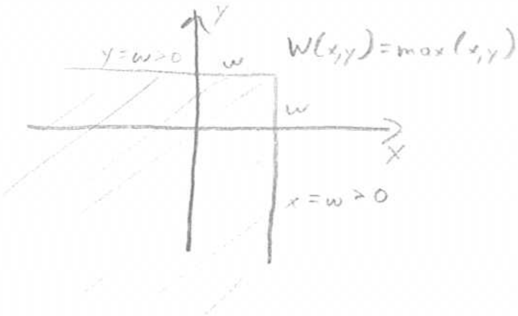
$$= \frac{1}{2} + 2v - 2v\ln(-4v)$$

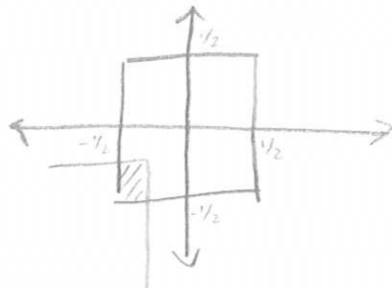
$$F_V(v) = \begin{cases} 0, & v \le -1/4 \\ \frac{1}{2} + 2v - 2v\ln(|4v|), & -1/4 < v < \frac{1}{4} \\ 1, & 1/4 \le v \end{cases}$$

3.18 cont...)

e) $W: \mathbb{R}^2 \to \mathbb{R}$

$W(x,y) = \max(x,y)$





$$F_W(w) = \begin{cases} 0, & w < -1/2 \\ (w+\frac{1}{2})^2, & -1/2 \le w \le 1/2 \\ 1, & 1/2 < w \end{cases}$$

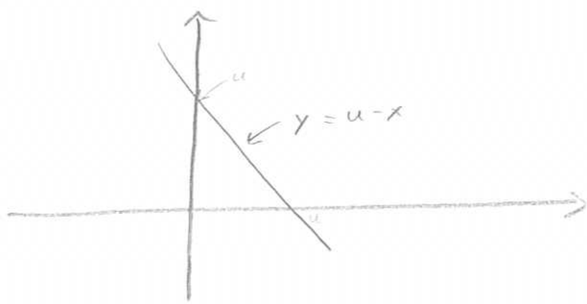
3.20)

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$$f_X(x) = f_Y(y) = \lambda e^{-\lambda x} \; ; \; x \geq 0$$

a) Find the pdf of $U = X + Y$.

Let's take a look at the CDF, $F_U(u) = P_r(U \leq u)$



$$P_r(U \leq u) = \int_0^u \int_0^{u-x} f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_0^u \int_0^{u-x} (\lambda e^{-\lambda x})(\lambda e^{-\lambda y})\,dy\,dx$$

$$= \int_0^u \lambda e^{-\lambda x} \left[\int_0^{u-x} \lambda e^{-\lambda y}\,dy \right] dx$$

$$= \int_0^u \lambda e^{-\lambda x} \left[-e^{-\lambda y} \right]_0^{u-x} dx$$

$$= \int_0^u \lambda e^{-\lambda x} \left[-e^{-\lambda(u-x)} + 1 \right] dx$$

$$= \int_0^u \left(\left[-\lambda e^{-\lambda x - \lambda u + \lambda x} \right] + \lambda e^{-\lambda x} \right) dx$$

$$= \int_0^u (-\lambda e^{-\lambda u} + \lambda e^{-\lambda x})\,dx$$

$$= \left[-\lambda e^{-\lambda u} x \right]_0^u + \left[-e^{-\lambda x} \right]_0^u$$

$$= -\lambda e^{-\lambda u} u + \left[1 - e^{-\lambda u} \right]$$

$$= 1 - (1 + \lambda u) e^{-\lambda u}$$

3.20 cont...)

cont... a) Since $F_U(u) = \begin{cases} 0 & , u < 0 \\ 1 - (1+\lambda u)e^{-\lambda u} & , u \geq 0 \end{cases}$

$$f_U(u) = \frac{dF_U(u)}{du} = -\left[(1+\lambda u)(-\lambda)e^{-\lambda u} + \lambda e^{-\lambda u}\right]$$

$$= (\lambda + \lambda^2 u)e^{-\lambda u} - \lambda e^{-\lambda u}$$

$$= \underline{\underline{\lambda^2 u e^{-\lambda u}}}, \quad u \geq 0$$

b) (W, V), $W = \min(X, Y)$, $V = \max(X, Y)$

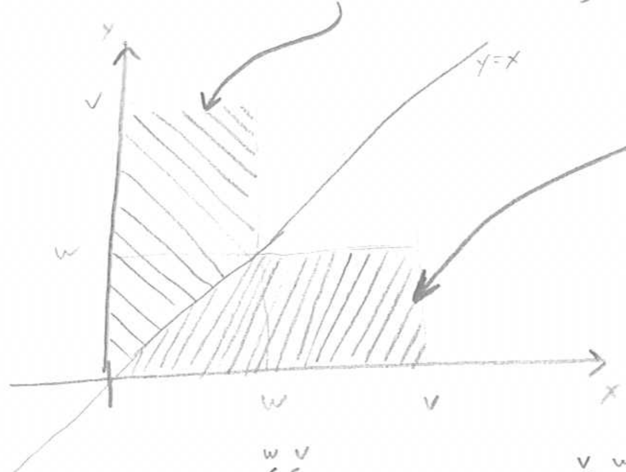
We have $W \leq V$. Thus, $f_{W,V}(w,v) = 0$ for $w > v$.

For $w \leq v$,

$$F_{W,V}(w,v) = P_r(W \leq w, V \leq v)$$

$$= P_r(\min(X,Y) \leq w, \max(X,Y) \leq v)$$

$$= P(Y \geq X, X \leq w, Y \leq v) + P(Y < X, Y \leq w, X \leq v)$$



$$F_{W,V}(w,v) = \int_0^w \int_0^v f_{X,Y}(x,y)\,dy\,dx + \int_w^v \int_0^w f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_0^w \int_0^v (\lambda e^{-\lambda x})(\lambda e^{-\lambda y})\,dy\,dx + \int_w^v \int_0^w (\lambda e^{-\lambda x})(\lambda e^{-\lambda y})\,dy\,dx$$

$$= \left[-e^{-\lambda y}\right]_0^v \left[-e^{-\lambda x}\right]_0^w + \left[-e^{-\lambda y}\right]_0^w \left[-e^{-\lambda x}\right]_w^v$$

$$= \left[1 - e^{-\lambda v}\right]\left[1 - e^{-\lambda w}\right] + \left[1 - e^{-\lambda w}\right]\left[e^{-\lambda w} - e^{-\lambda v}\right]$$

3.20 cont...)

b cont...)

$$F_{W,V}(w,v) = \left(1 - e^{-\lambda v} \cancel{- e^{-\lambda w}} + e^{-\lambda(v+w)}\right) + \left(\cancel{e^{-\lambda w}} - e^{-\lambda v} - e^{-2\lambda w} + e^{-\lambda(v+w)}\right)$$

$$= 1 - 2e^{-\lambda v} - e^{-2\lambda w} + 2e^{-\lambda(v+w)}$$

$$f_{W,V}(w,v) = \frac{\partial^2 F_{W,V}(w,v)}{\partial v\, \partial w} = 2\lambda e^{-\lambda v} + 2(-\lambda)\, e^{-\lambda(v+w)}$$

$$= 2\lambda^2 e^{-\lambda(v+w)}$$

$$f_{W,V}(w,v) = \begin{cases} 2\lambda^2 e^{-\lambda(v+w)} & , \ 0 \le w \le v \\ 0 & , \text{otherwise} \end{cases}$$

3.21) (X, Y) with $P_{X,Y}$

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$$f_X(x) = f_Y(y) = \frac{e^{-r^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}$$

$$R = \sqrt{X^2 + Y^2}$$

$$\Theta = \tan^{-1}(Y/X)$$

Find joint pdf of (R, Θ)

$$F_{R,\Theta}(r, \theta) = P_r(R \le r, \Theta \le \theta)$$

$$= \iint_{x,y:\sqrt{x^2+y^2} \le r,\ \tan^{-1}(y/x) \le \theta} f_{X,Y}(x,y)\,dx\,dy$$

$$= \iint_{x,y:\sqrt{x^2+y^2} \le r,\ \tan^{-1}(y/x) \le \theta} \frac{e^{-(x^2+y^2)/2\sigma^2}}{2\pi\sigma^2}\,dx\,dy$$

Change to polar coordinates

$$\rho^2 = x^2 + y^2 \qquad dx\,dy = \rho\,d\rho\,d\phi$$

$$\phi = \tan^{-1}(y/x)$$

$$F_{R,\Theta}(r,\theta) = \iint_{\rho,\phi:\rho \le r,\ -\pi \le \phi \le \theta} \frac{e^{-\rho^2/2\sigma^2}}{2\pi\sigma^2}\,\rho\,d\rho\,d\phi$$

$$= (\theta + \pi) \int_0^r \frac{e^{-\frac{\rho^2}{2\sigma^2}}}{2\pi\sigma^2}\,\rho\,d\rho \ ; \quad \theta \in [-\pi, \pi],\ r \ge 0$$

$$f_{R,\Theta}(r,\theta) = \frac{\partial^2}{\partial r\,\partial\theta} F_{R,\Theta}(r,\theta)$$

$$= \frac{1}{2\pi} \frac{e^{-r^2/2\sigma^2}}{\sigma^2}\,r \quad ; \theta \in [-\pi, \pi],\ r \ge 0$$

$$\Rightarrow f_\Theta(\theta) = \frac{1}{2\pi} \ ; \theta \in [-\pi, \pi] \qquad f_R(r) = \frac{e^{-r^2/2\sigma^2}}{\sigma^2}\,r \ ; r \ge 0$$

Θ and R are independent.

3.22) $(\Omega, \mathcal{F}, P)$

$\omega = (\omega_0, \ldots, \omega_{k-1})$ where ω_i is 0 or 1.

P is pmf with probability of $\frac{1}{2^8}$ to each of the 2^8 elements.

a) $g(\omega) = \sum_{i=0}^{k-1} \omega_i$

Binomial with $n = 8$, $p = 0.5$

Random variable G takes values in $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

$$P_G(g) = P(G = g) = P(\{\omega : \sum_{i=0}^{7} \omega_i = g\})$$

$$= \underline{\binom{8}{g} \cdot \frac{1}{2^8}}$$

b) $X(\omega) = 1$ if there are an even number of 1s in ω and 0 otherwise.

$$P_X(1) = P(X = 1) = P(\{\omega : \sum_{i=0}^{7} \omega_i = 2k \text{ "even"}\})$$

$$= P_G(0) + P_G(2) + P_G(4) + P_G(6) + P_G(8)$$

$$= \frac{1}{2^8} \cdot \left(\binom{8}{0} + \binom{8}{2} + \binom{8}{4} + \binom{8}{6} + \binom{8}{8}\right)$$

$$= \frac{1}{2}$$

$$P_X(x) = \begin{cases} 1/2, & x = 0 \\ 1/2, & x = 1 \end{cases}$$

c) $Y(\omega) = \omega_j$, i.e. the value of the j^{th} coordinate of ω.

$$Y = \begin{cases} 1, & \text{if } \omega_j = 1 \\ 0, & \text{if } \omega_j = 0 \end{cases}$$

$$P_Y(1) = P(Y = 1) = P(\{\omega : \omega_j = 1\})$$

$$= \frac{(\text{\# outcomes such that } \omega_j = 1 \text{ and } \omega_i = 0 \text{ or } 1,\ i \neq j)}{\text{\# total outcomes}}$$

$$= \frac{\sum_{n=0}^{7} \binom{7}{n}}{2^8} = \frac{2^7}{2^8} = \frac{1}{2}$$

$$P_Y(y) = \begin{cases} 1/2, & y = 1 \\ 1/2, & y = 0 \end{cases}$$

d) $Z(\omega) = \max_i(\omega_i)$

$$Z = \begin{cases} 1, & \text{if } \max_i(\omega_i) = 1 \\ 0, & \text{if } \max_i(\omega_i) = 0 \end{cases}$$

$$p_Z(0) = P(Z=0) = P(\{\omega_i : \max_i(\omega_i) = 0\})$$

$$= P(\{0,0,0,0,0,0,0,0\}) = \frac{1}{2^8}$$

$$p_Z(1) = 1 - p_Z(0) = 1 - \frac{1}{2^8}$$

$$p_Z(z) = \begin{cases} 255/256, & z = 1 \\ 1/256, & z = 0 \end{cases}$$

e) $V(\omega) = g(\omega)X(\omega)$

V can take on values $\{0, 2, 4, 6, 8\}$

$$p_V(2) = p_G(2) = \binom{8}{2} \cdot \frac{1}{2^8} = \frac{28}{256}$$

$$p_V(4) = p_G(4) = \binom{8}{4} \cdot \frac{1}{2^8} = \frac{70}{256}$$

$$p_V(6) = p_G(6) = \binom{8}{6} \cdot \frac{1}{2^8} = \frac{28}{256}$$

$$p_V(8) = p_G(8) = \binom{8}{8} \cdot \frac{1}{2^8} = \frac{1}{256}$$

$$p_V(0) = 1 - (p_V(2) + p_V(4) + p_V(6) + p_V(8))$$

$$p_V(v) = \begin{cases} 129/256, & v = 0 \\ 28/256, & v = 2 \\ 70/256, & v = 4 \\ 28/256, & v = 6 \\ 1/256, & v = 8 \end{cases}$$

3.23) $(X_0, X_1, \dots, X_N)$ is iid random vector with marginal pdf's

$$f_{X_n}(\alpha) = \begin{cases} 1 & 0 \le \alpha < 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{Uniform R V.}$$

a) $U = X_0^{\,2}$

$$F_{X_0}(x) = P_r(X_0 \le x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & 1 < x \end{cases}$$

$$F_U(u) = P_r(U \le u) = P_r(X_0^{\,2} \le u) = P_r(\sqrt{u} \le X_0 \le \sqrt{u})$$

$$= F_{X_0}(\sqrt{u}) = \begin{cases} 0, & u < 0 \\ \sqrt{u}, & 0 \le u \le 1 \\ 1, & 1 < u \end{cases}$$

$$f_U(u) = \frac{dF_U(u)}{du} = \begin{cases} \frac{1}{2}u^{-1/2}, & 0 \le u \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$V = \max(X_1, X_2, X_3, X_4)$ Range: $[0, 1]$

$$F_V(v) = P_r(V \le v) = P_r(\max(X_1, X_2, X_3, X_4) \le v)$$

$$= P_r(X_1 \le v, X_2 \le v, X_3 \le v, X_4 \le v)$$

Because iid

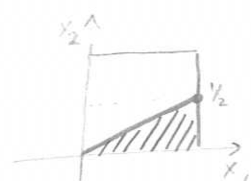
$$= P_r(X_1 \le v)\,P_r(X_2 \le v)\,P_r(X_3 \le v)\,P_r(X_4 \le v)$$

$$= F_{X_1}(v)\,F_{X_2}(v)\,F_{X_3}(v)\,F_{X_4}(v)$$

$$= \begin{cases} 0, & v < 0 \\ v^4, & 0 \le v \le 1 \\ 1, & 1 < v \end{cases}$$

$$f_V(v) = \begin{cases} 4v^3, & 0 \le v \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$W = \begin{cases} 1 & \text{if } X_1 \ge 2X_2 \\ 0 & \text{otherwise} \end{cases}$$



$$P(X_1 \ge 2X_2) = \iint_{\text{shaded area}} f_{X_1 X_2}(x_1, x_2)\,dx_1\,dx_2 = 1/4$$

pmf: $P_W(w) = \begin{cases} 1/4, & \text{if } w = 1 \\ 3/4, & \text{if } w = 0 \end{cases}$

b) $Y_n = X_n + X_{n-1}$; $n = 1, \dots, N$

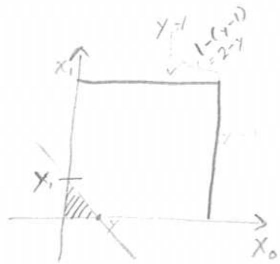
Range of Y_n : $[0, N]$

$$F_{X_i}(x) = P_r(X_i \le x) = \begin{cases} 0 , & x < 0 \\ x , & 0 \le x \le 1 \\ 1 , & 1 < x \end{cases}$$

cdf of Y_n

$$F_{Y_n}(y) = P_r(Y_n \le y) = P_r(X_n + X_{n-1} \le y)$$

Since the X_i are iid, this can be done for a representative X_0 and X_1.



$y = x_0 + x_1$

$x_1 = y - x_0$

$y = x_0 + 1$

$x_0 = y - 1$

$y = 1 + x_1$

$x_1 = y - 1$

This is thus exactly the same as 3,18 c) except shifted by ½

$$P(Y \le y) = \begin{cases} 0 , & y < 0 \\ \frac{1}{2} y^2 , & 0 \le y \le 1 \\ 1 - \frac{1}{2}(2-y)^2 , & 1 < y \le 2 \\ 1 , & 2 < y \end{cases}$$

3.35) $\vec{X} = (X_0, \ldots, X_{k-1})$ is iid with marginal pmf

$$P_{X_i}(\ell) = P_X(\ell) = \begin{cases} p & \text{if } \ell = 1 \\ 1-p & \text{if } \ell = 0 \end{cases} \quad \text{for all } i.$$

a) Find pmf of $Y = \prod_{i=0}^{k-1} X_i$

$$P_Y(1) = P_r(\vec{X} = (1, \ldots, 1))$$
$$= \prod_{i=0}^{k-1} P_{X_i}(1) \quad \text{because iid}$$
$$= p^k$$
$$P_Y(0) = 1 - p^k$$

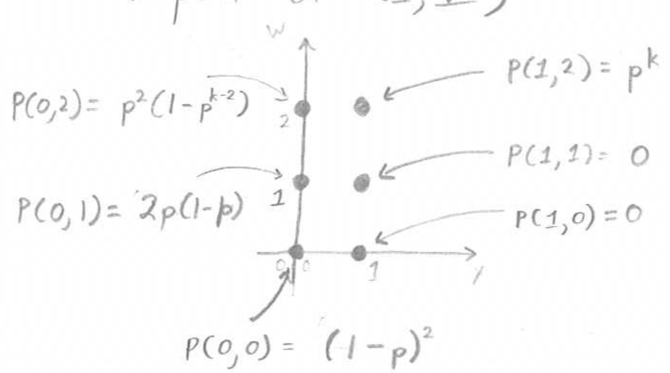
$$P_Y(y) = \begin{cases} 1-p^k, & \text{if } y = 0 \\ p^k, & \text{if } y = 1 \\ 0, & \text{otherwise} \end{cases}$$

b) $W = X_0 + X_{k-1}$

W range is $0, 1, 2$.

$$P_W(w) = \begin{cases} (1-p)^2, & \text{if } w = 0 \\ 2p(1-p), & \text{if } w = 1 \\ p^2, & \text{if } w = 2 \\ 0, & \text{otherwise} \end{cases}$$

c) pmf of (Y, W)



$$\sum P(Y, W) =$$
$$p^k + p^2(1-p^{k-2}) + 2p(1-p) + (1-p)^2$$
$$= \cancel{p^k} + \cancel{p^2} - \cancel{p^k} + \cancel{2p} - \cancel{2p^2} + 1 - \cancel{2p} + \cancel{p^2}$$
$$= 1.$$

3.38) $\{X_n\}: \; p_X(1) = p_X(-1) = \frac{1}{2}$

$\{Y_n\}: \; N(0,1) \quad \begin{matrix} m=0 \\ \sigma=1 \end{matrix}$

$\{Z_n = X_n + Y_n\}$

a) Find pdf of Z_n

$$F_{Z_n}(z) = P(X+Y \le z)$$
$$= P(X_n + Y_n \le z \mid X_n = +1)\, p_X(+1) + P(X_n + Y_n \le z \mid X_n = -1)\, p_X(-1)$$
$$= P(1 + Y_n \le z)\, p_X(+1) + P(-1 + Y_n \le z)\, p_X(-1)$$
$$= P(Y_n \le z-1)\, p_X(+1) + P(Y_n \le z+1)\, p_X(-1)$$
$$= \frac{1}{2}\int_{-\infty}^{z-1} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy + \frac{1}{2}\int_{-\infty}^{z+1} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy$$

$$f_{Z_n}(z) = \frac{d}{dz} F_{Z_n}(z)$$
$$= \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-(z-1)^2/2} + \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-(z+1)^2/2}$$
$$= \frac{1}{2\sqrt{2\pi}}\left(e^{-(z^2-2z+1)/2} + e^{-(z^2+2z+1)/2}\right)$$
$$= \frac{e^{-(z^2+1)/2}}{2\sqrt{2\pi}}\left(e^{z} + e^{-z}\right)$$

b) $\{R_n = \text{sgn}(Z_n)\}$

pmf of R_n

We see that $f_{Z_n}(z) = f_{Z_n}(-z)$. Thus, $P(Z<0) = P(Z>0) = \frac{1}{2}$

$P_r(R_n = 1) = P_r(Z_n \ge 0) = \frac{1}{2}$

$P_r(R_n = -1) = P_r(Z_n < 0) = 1/2$

$$p_{R_n}(r) = \begin{cases} 1/2, & r = 1 \\ 1/2, & r = -1 \\ 0, & \text{otherwise} \end{cases}$$

$$P_r(R_n = X_n) = P(R_n = X_n \mid X_n = 1)\, p_X(+1) + P(R_n = X_n \mid X_n = -1)\, p_X(-1)$$
$$= P(+1 + Y_n \ge 0)\, p_X(+1) + P(-1 + Y_n < 0)\, p_X(-1)$$
$$= P(Y_n \ge -1)\, p_X(+1) + P(Y_n < 1)\, p_X(-1)$$

3.38 cont...)

$$Pr(R_n = X_n) = P(Y_n < 1)\left(p_X(+1) + p_X(-1)\right) = P(Y_n < 1)$$

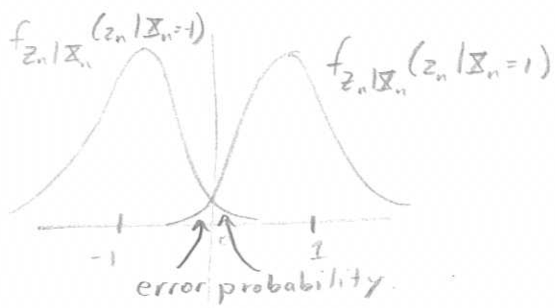
From standard normal table

$$Pr(R_n = X_n) = 0.8413$$

c) Is this detector optimal?

If $\hat{X}_n$ is our approximation for X_n then we want to minimize $Pr(\hat{X}_n \neq X_n)$ or maximize $Pr(\hat{X}_n = X_n)$. The detector, to be optimal, needs $\hat{X}_n = 1$ if

$$Pr(X_n = 1 \mid Z_n = z_n) > Pr(X_n = -1 \mid Z_n = z_n)$$



$$\frac{1}{\sqrt{2\pi}} e^{-\frac{(z_n-1)^2}{2}} > \frac{1}{\sqrt{2\pi}} e^{-\frac{(z_n+1)^2}{2}}$$

$$(z_n-1)^2 < (z_n+1)^2$$

$$|z_n-1| < |z_n+1|$$

$$z_n > 0$$

Thus, this detector $\{R_n = \text{sgn}(Z_n)\}$ is optimal.

3.39) $Y(t) = X \cos(2\pi f_0 t)$

$Y(t) = a(t) X$

$P(Y(t) \leq y) = P(a(t) X \leq y)$

$= P(X \leq \frac{y}{a(t)})$

Since X is Gaussian

$P(X \leq x) = \Phi\left(\frac{x-m}{\sigma}\right)$

$P(X \leq \frac{y}{a(t)}) = \Phi\left(\frac{\frac{y}{a(t)} - m}{\sigma}\right)$

$= \Phi\left(\frac{y - a(t)m}{a(t)\sigma}\right)$

$N(a(t)m, [a(t)\sigma]^2)$

So, the marginal pdf

$f_{Y(t)}$ is $N\left(\cos(2\pi f_0 t)\, m,\ \cos^2(2\pi f_0 t)\, \sigma^2\right)$

Where m is the mean of $f_X(x)$ and σ^2 is the variance of $f_X(x)$.

3.43)

Prove the following facts about characteristic functions

(a) $|M_X(ju)| \leq 1$

$$M_X(ju) = E[e^{juX}] = \begin{cases} \sum_x p_X(x) e^{jux} & \text{discrete} \\ \int f_X(x) e^{jux}\, dx & \text{continuous} \end{cases}$$

$$|M_X(ju)| = \begin{cases} \left|\sum_x p_X(x) e^{jux}\right| \leq \left|\sum_x p_X(x) |e^{jux}|\right| = \left|\sum_x p_X(x)\right| = 1 \\ \left|\int f_X(x) e^{jux}\right| \leq \left|\int f_X(x) |e^{jux}|\right| = \left|\int f_X(x)\right| = 1 \end{cases}$$

$\therefore |M_X(ju)| \leq 1$

b) $M_X(0) = 1$

$$M_X(0) = E[e^{0X}] = E[1] = \begin{cases} \sum_x p_X(x) \\ \int f_X(x)\, dx \end{cases} = \underline{1.}$$

c) $|M_X(ju)| \leq M_X(0) = 1$

From a)

$|M_X(ju)| \leq 1$

From b)

$M_X(0) = 1$

$\therefore |M_X(ju)| \leq M_X(0) = 1$

d) X has characteristic function $M_X(ju)$

$Y = X + c$

$$M_Y(ju) = E[e^{juY}] = E[e^{ju(X+c)}] = \begin{cases} \sum_x p_X(x) e^{ju(x+c)} \\ \int f_X(x) e^{ju(x+c)} \end{cases}$$

$$= \begin{cases} e^{juc} \sum_x p_X(x) e^{jux} \\ e^{juc} \int f_X(x) e^{jux} \end{cases} = e^{juc} E[e^{juX}] = e^{juc} M_X(ju)$$

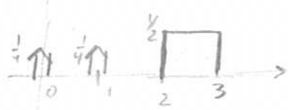
3.48) Two coins $\quad$ U is uniform over $[0, 1)$.

If first coin is "heads" $p = 0.5$

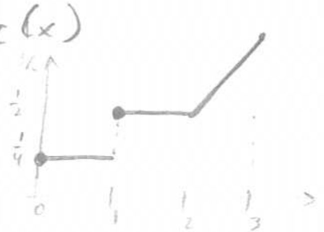
$$\mathbb{X} = \begin{cases} 1 & \text{if 2nd coin is "heads"} \\ 0 & \end{cases}$$

If first coin is "tails" $p = 0.5$, then $\mathbb{X} = U + 2$

$f_{\mathbb{X}}(x)$



a) $F_{\mathbb{X}}(x)$



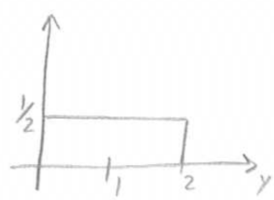
$$F_{\mathbb{X}}(x) = \begin{cases} 0, & x < 0 \\ 1/4, & 0 \le x < 1 \\ 1/2, & 1 \le x < 2 \\ \frac{1}{2} + \frac{1}{2}(x-2), & 2 \le x < 3 \\ 1, & 3 \le x \end{cases}$$

b) $Pr\left(\frac{1}{2} \le \mathbb{X} \le 2\right) = F_{\mathbb{X}}(2) - F_{\mathbb{X}}(1/2) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

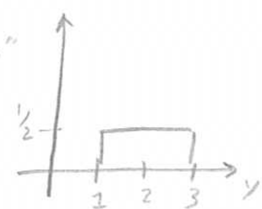
c)

$$\mathbb{Y} = \begin{cases} 2U & \text{if 1st coin is "heads"} \\ 2U + 1 & \text{otherwise} \end{cases}$$

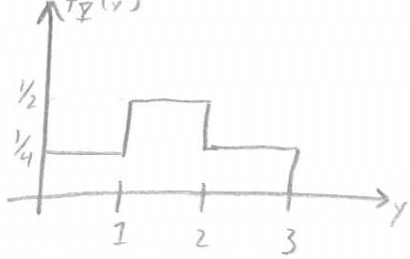
$f_{\mathbb{Y}|\text{"Heads"}}$



$f_{\mathbb{Y}|\text{"tails"}}$



$f_{\mathbb{Y}}(y) = f_{\mathbb{Y}|\text{"heads"}}(y|\text{"heads"})\,P(\text{"heads"}) + f_{\mathbb{Y}|\text{"tails"}}(y|\text{"tails"})\,P(\text{"tails"})$

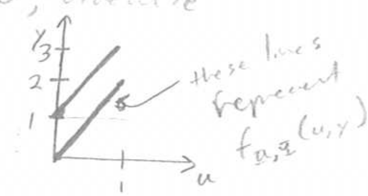


d) Design an optimal detection rule to estimate U if you are given only Y. What is the probability of error?

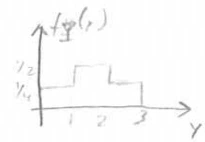
We propose an estimator $\hat{U}(y)$. We want to minimize $\Pr(\hat{U} \neq U)$ or maximize $\Pr(\hat{U} = U)$.

$$Y = \begin{cases} 2U & \text{if first coin is "heads"} \\ 2U+1 & \text{otherwise} \end{cases}$$

$$P_{Y|U}(y|u) = \begin{cases} 1/2, & \text{if } y = 2u \\ 1/2, & \text{if } y = 2u+1 \end{cases} \qquad f_U(u) = \begin{cases} 1, & \text{if } 0 \le u < 1 \\ 0, & \text{otherwise} \end{cases}$$



$$f_{U|Y}(u|y) = \frac{P_{Y|U}(y|u)\, f_U(u)}{f_Y(y)} = \frac{f_{U,Y}(u,y)}{f_Y(y)} =$$



$$f_{U|Y}(u|y) = \begin{cases} 1 & \text{if } u = \frac{y}{2}, & 0 \le y < 1 \\ 0 & \text{if } u \neq \frac{y}{2}, & 0 \le y < 1 \\ 1/2 & \text{if } u = y/2, & 1 \le y < 2 \\ 1/2 & \text{if } u = \frac{y-1}{2}, & 1 \le y < 2 \\ 1 & \text{if } u = \frac{y-1}{2}, & 2 \le y < 3 \\ 0 & \text{if } u \neq \frac{y-1}{2}, & 2 \le y < 3 \end{cases}$$

$$\hat{U}(y) = \arg\max_u f_{U|Y}(u|y)$$

$$= \begin{cases} \frac{y}{2}, & 0 \le y < 1 \\ y/2, & 1 \le y < 2 \\ \frac{y-1}{2}, & 2 \le y < 3 \end{cases} \quad \text{or equally optimal} \quad = \begin{cases} \frac{y}{2}, & 0 \le y < 1 \\ \frac{y-1}{2}, & 1 \le y < 2 \\ \frac{y-1}{2}, & 2 \le y < 3 \end{cases}$$

$$P_e = \Pr(\hat{U} \neq U) = \Pr(\hat{U} \neq u \mid U = u)$$
$$= \Pr\left(\frac{Y}{2} \neq U \mid 1 \le y < 2\right) \Pr(1 \le y < 2)$$
$$= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

3.48 e) i) Z

$P_Z(1) = p$, given U

$$Z = \begin{cases} 1, & 0 \le U < p \\ 0, & \text{otherwise} \end{cases}$$

This is how computers frequently do this calculation.

ii) Generate a continuous, uniformly distributed variable given Z.

if we have $p = 0.5$, and an infinite family of Z_n, for $n = 0, 1, \ldots$

we can generate a uniform random variable

$$U = \sum_{n=0}^{\infty} b_n(r)\, 2^{-n+1}$$

for $b_n(r) = Z_n$

3.55) $\{X_n ; n = 0, 1, 2, \dots\}$ iid

$$P_{X_n}(k) = \begin{cases} p & \text{if } k = 1 \\ 1-p & \text{if } k = 0 \end{cases} \quad \text{for all } n$$

$\{W_n ; n = 0, 1, \dots\}$ iid

$$P_{W_n}(k) = \begin{cases} \epsilon & \text{if } k = 1 \\ 1-\epsilon & \text{if } k = 0 \end{cases}$$

$$Y_n = X_n \oplus W_n$$

a)

$$1 - p - \epsilon + p\epsilon + p\epsilon$$

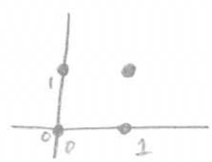
$$P_{Y_n}(k) = \begin{cases} (1-p)(1-\epsilon) + p\epsilon & , \text{if } k = 0 \\ p(1-\epsilon) + (1-p)\epsilon & , \text{if } k = 1 \end{cases} = \begin{cases} 1 - (p + \epsilon - 2p\epsilon) & , \text{if } k = 0 \\ p + \epsilon - 2p\epsilon & , \text{if } k = 1 \end{cases}$$

$Y_n = 1$ if $X_n = 0,\ W_n = 1 \quad (1-p)\epsilon$

$X_n = 1,\ W_n = 0 \quad p(1-\epsilon)$

$Y_n = 0$ if $X_n = 0,\ W_n = 0 \quad (1-p)(1-\epsilon)$

$X_n = 1,\ W_n = 1 \quad p\epsilon$



b) $\{Y_n\}$ is Bernoulli, or i.i.d. because there is no memory in the system.

c)

$$P_{Y_n | X_n}(j | k) = \begin{cases} \epsilon & , j = k \\ 1-\epsilon & , j \neq k \end{cases}$$

d)

$$P_{X_n | Y_n}(k | j) = \begin{cases} \dfrac{(1-p)(1-\epsilon)}{(1-p)(1-\epsilon) + p\epsilon} \text{ if } k = 0, j = 0 & \dfrac{(1-p)\epsilon}{(1-p)\epsilon + p(1-\epsilon)} \text{ if } k = 0, j = 1 \\ \dfrac{p\epsilon}{(1-p)(1-\epsilon) + p\epsilon} \text{ if } k = 1, j = 0 & \dfrac{p(1-\epsilon)}{(1-p)\epsilon + p(1-\epsilon)} \text{ if } k = 1, j = 1 \end{cases}$$

e)

$$P_r(Y_n \neq X_n) = P_r(Y = 1 | X = 0) P_r(X = 0) + P_r(Y = 0 | X = 1) P_r(X = 1)$$

$$= (\epsilon)(1-p) + (\epsilon)(p) = \epsilon$$

f) Estimate $\hat{X}(j)$

$P_e = P_r(\hat{X}(Y_n) \neq X_n)$. Consider part d) $P_{X_n|Y_n}(k|j)$.

If $j=0$, we decide $\hat{X}(j)=0$ if $(1-p)(1-\epsilon) > p\epsilon$, otherwise $\hat{X}(j)=1$.

$$1-p-\epsilon+p\epsilon > p\epsilon$$
$$1 > p+\epsilon$$

If $j=1$, we decide $\hat{X}(j)=1$ if $p(1-\epsilon) > (1-p)\epsilon$, otherwise $\hat{X}(j)=0$.

$$p-p\epsilon > \epsilon - p\epsilon$$
$$p > \epsilon$$

Using this decision rule,

$$P_e = P_r(\hat{X}(Y_n) \neq X_n)$$
$$= P_r(\hat{X}(Y_n)=1 \mid X=0)P_r(X=0) + P_r(\hat{X}(Y_n)=0 \mid X=1)P_r(X=1)$$
$$= \left[P_r(\hat{X}(Y_n)=1 \mid X=0, Y=0) + P_r(\hat{X}(Y_n)=1 \mid X=0, Y=1)\right]P_r(X=0)$$
$$+ \left[P_r(\hat{X}(Y_n)=0 \mid X=1, Y=0) + P_r(\hat{X}(Y_n)=0 \mid X=1, Y=1)\right]P_r(X=1)$$
$$= P_r(\hat{X}(Y_n)=1 \mid X=0, Y=0)\underbrace{P_r(X=0, Y=0)}_{(1-p)(1-\epsilon)} + P_r(\hat{X}(Y_n)=1 \mid X=0, Y=1)\underbrace{P_r(X=0, Y=1)}_{(1-p)\epsilon}$$
$$+ P_r(\hat{X}(Y_n)=0 \mid X=1, Y=0)\underbrace{P_r(X=1, Y=0)}_{p\epsilon} + P_r(\hat{X}(Y_n)=0 \mid X=1, Y=1)\underbrace{P_r(X=1, Y=1)}_{p(1-\epsilon)}$$

If $1 > p+\epsilon,\ p > \epsilon$: $P_e = p\epsilon + (1-p)\epsilon$

$1 > p+\epsilon,\ p \leq \epsilon$: $P_e = p\epsilon + p(1-\epsilon)$

$1 \leq p+\epsilon,\ p > \epsilon$: $P_e = (1-p)(1-\epsilon) + (1-p)\epsilon$

$1 \leq p+\epsilon,\ p \leq \epsilon$: $P_e = (1-p)(1-\epsilon) + p(1-\epsilon)$