ECEn 670

Homework Problem Set 4

Due at beginning of class, Thursday, October 22, 2009

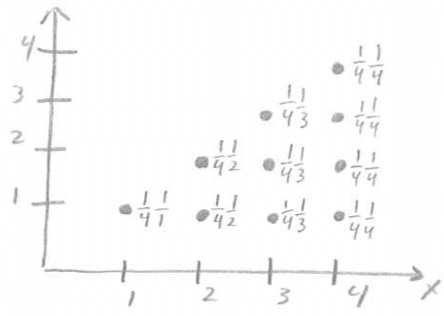
Problems are from *An Introduction to Statistical Signal Processing* by Gray and Davisson unless otherwise specified.

1. 4.1
2. 4.3
3. 4.5
4. 4.6
5. 4.7
6. 4.19
7. 4.22
8. 4.26
9. 4.29
10. 4.30
11. 4.31

4.1) $P_X(k) = \begin{cases} 1/4, & x = 1,2,3,4 \\ 0, & \text{otherwise} \end{cases}$

$P_{N|X}(n|k) = \frac{1}{k}\ ; n = 1, 2, \ldots, k.$

a) $P_{X,N}(x,n) = P_{N|X}(n|x)\, P_X(x)$



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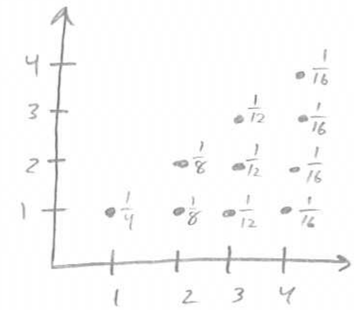
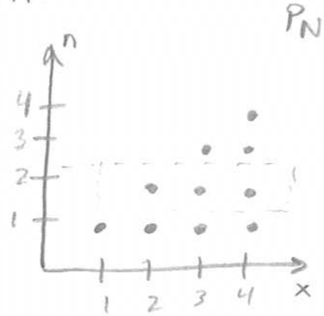


b) $P_N(n) = \begin{cases} \frac{1}{4}+\frac{1}{8}+\frac{1}{12}+\frac{1}{16}, & n=1 \\ \frac{1}{8}+\frac{1}{12}+\frac{1}{16}, & n=2 \\ \frac{1}{12}+\frac{1}{16}, & n=3 \\ \frac{1}{16}, & n=4 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{25}{48}, & n=1 \\ \frac{13}{48}, & n=2 \\ \frac{7}{48}, & n=3 \\ \frac{1}{16}, & n=4 \\ 0, & \text{otherwise} \end{cases}$

c) $P_{X|N}(x|n) = \dfrac{P_{X,N}(x,n)}{P_N(n)}$



$P_N(2) = \frac{13}{48}$

$P_{X|N}(x|2) = \begin{cases} \frac{1/8}{13/48}, & x=2 \\ \frac{1/12}{13/48}, & x=3 \\ \frac{1/16}{13/48}, & x=4 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 6/13, & x=2 \\ 4/13, & x=3 \\ 3/13, & x=4 \\ 0, & \text{otherwise} \end{cases}$

d) $P_{X|A} = \dfrac{P_{X,A}(x)}{P(A)} = \dfrac{P_{X,A}(x)}{\frac{1}{12}+\frac{1}{16}+\frac{1}{8}+\frac{1}{12}+\frac{1}{16}} = \dfrac{P_{X,A}(x)}{\frac{4+3+6+4+3}{48}} = \dfrac{P_{X,A}(x)}{\frac{20}{48}} = \dfrac{P_{X,A}(x)}{5/12}$

$P_{X|A}(x) = \begin{cases} \frac{1/8}{5/12}, & x=2 \\ \frac{1/12+1/12}{5/12}, & x=3 \\ \frac{1/16+1/16}{5/12}, & x=4 \end{cases} = \begin{cases} 3/10, & x=2 \\ 2/5, & x=3 \\ 3/10, & x=4 \\ 0, & \text{otherwise} \end{cases}$

$E[X|A] = 2\left(\frac{3}{10}\right) + 3\left(\frac{2}{5}\right) + 4\left(\frac{3}{10}\right) =$
$= \frac{3}{5} + \frac{6}{5} + \frac{6}{5} = \frac{15}{5} = \boxed{3}$

$E[X^2|A] = 4\left(\frac{3}{10}\right) + 9\left(\frac{2}{5}\right) + 16\left(\frac{3}{10}\right)$
$= \frac{6}{5} + \frac{18}{5} + \frac{24}{5} = \frac{48}{5}$

$\text{var}(X^2|A) = E[X^2|A] - (E[X|A])^2 = \frac{48}{5} - 9$
$= \frac{48}{5} - \frac{45}{5} = \boxed{3/5}$

e) $E[C] = 3$ where C is cost of a book

$T = \sum_{i=1}^{n} C_i$ where T is total cost and n is books bought

$E[T] \stackrel{*}{=} E[N]E[C] = \frac{7}{4} \cdot 3 = \boxed{\frac{21}{4}}$

$\hookrightarrow E[N] = (1)\frac{25}{48} + (2)\left(\frac{13}{48}\right) + (3)\left(\frac{7}{48}\right) + (4)\left(\frac{3}{48}\right) = \frac{25}{48} + \frac{26}{48} + \frac{21}{48} + \frac{12}{48} = \frac{84}{48} = \frac{21}{12} = \frac{7}{4}$

4.3) $p_Z(k) = C \dfrac{a^k}{(1+a)^{k+1}}$, $k = 0, 1, \ldots$

$p_Z(k) \geq 0, \; \forall k \in \{0, 1, 2, \ldots\} \Rightarrow C > 0$

$$\sum_{k=0}^{\infty} p_Z(k) = 1 = \sum_{k=0}^{\infty} C \frac{a^k}{(1+a)^{k+1}}$$

$$= C \sum_{k=0}^{\infty} \frac{a^k}{(1+a)^{k+1}}$$

$$= \frac{C}{1+a} \sum_{k=0}^{\infty} \left(\frac{a}{1+a}\right)^k$$

$$= \frac{C}{1+a} \cdot \frac{1}{1 - \frac{a}{1+a}}$$

$$= \frac{C}{1+a} \cdot \frac{1+a}{1+a-a}$$

$$= C$$

$\therefore \underline{C = 1}$

$\therefore \; p_Z(k) = \dfrac{a^k}{(1+a)^{k+1}}$, $k = \{0, 1, 2, \ldots\}$

$$E[Z] = \sum_{k=0}^{\infty} k \cdot \frac{a^k}{(1+a)^{k+1}}$$

$$= \frac{a}{(1+a)^2} \sum_{k=0}^{\infty} k \cdot \left(\frac{a}{1+a}\right)^{k-1}$$

$$= \frac{a}{(1+a)^2} \cdot \frac{1}{\left(1 - \frac{a}{1+a}\right)^2}$$

$$= \frac{a}{(1+a)^2} \cdot \frac{(1+a)^2}{(1+a-a)^2} = a \qquad \therefore \; E[Z] = a$$

$$E[Z^2] = \sum_{k=0}^{\infty} k^2 \cdot \frac{a^k}{(1+a)^{k+1}}$$

$$= \frac{a^2}{(1+a)^3} \sum_{k=0}^{\infty} k^2 \cdot \left(\frac{a}{1+a}\right)^{k-2}$$

$$= \frac{a^2}{(1+a)^3} \cdot \left(\frac{2}{\left(1 - \frac{a}{1+a}\right)^3} + \frac{1+a}{a} \cdot \frac{1}{\left(1 - \frac{a}{1+a}\right)^2} \right)$$

$$= \frac{a^2}{(1+a)^3} \cdot \left(\frac{2(1+a)^3}{(1+a-a)^3} + \frac{(1+a)^3}{a(1+a-a)^2} \right) = 2a^2 + a$$

$$Var(Z) = E[Z^2] - (E[Z])^2 = 2a^2 + a - a^2 = \boxed{a^2 + a}$$

4.3 (cont...)

$$E[e^{juZ}] = \sum_{z=0}^{\infty} e^{juz}\, p_Z(z)$$

$$= \sum_{z=0}^{\infty} e^{juz} \cdot \frac{a^z}{(1+a)^{z+1}}$$

$$= \frac{1}{1+a} \sum_{z=0}^{\infty} \left(e^{ju} \cdot \frac{a}{1+a}\right)^z$$

Note that $\left| e^{ju} \cdot \frac{a}{1+a} \right| = \left| \frac{a}{1+a} \right| < 1$, so

$$E[e^{juZ}] = \frac{1}{1+a} \cdot \frac{1}{1 - e^{ju}\frac{a}{1+a}}$$

$$= \frac{1}{1+a} \cdot \frac{1+a}{1+a-ae^{ju}}$$

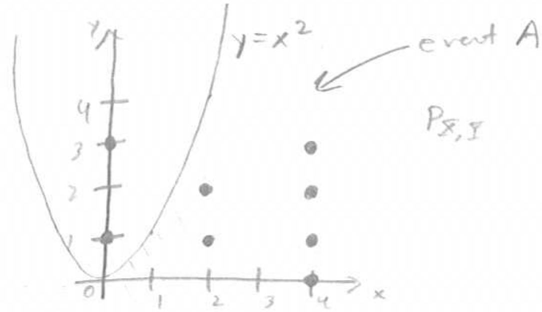
$$= \frac{1}{1+a-ae^{ju}}$$

$$= \frac{1}{1+a-a\cos(u) - ja\sin(u)}$$

$$= \frac{1+a-a\cos(u)+ja\sin(u)}{(1+a-a\cos(u))^2 + a^2\sin^2(u)}$$

$$= \frac{1+a(1-\cos(u))}{(1+a(1-\cos(u)))^2 + a^2\sin^2(u)} + j \cdot \frac{a\sin(u)}{(1+a(1-\cos(u)))^2 + a^2\sin^2(u)}$$

4.5. Given the setup of Problem 3.10:



a) $E[X] = \frac{1}{4}(0) + \frac{1}{4}(2) + \frac{1}{2}(4) = \frac{1}{2} + 2 = \frac{5}{2}$

$E[XY] = \frac{1}{8}(0+0+2+4+0+4+8+12)$

$= \frac{1}{8}(30) = \frac{30}{8} = \frac{15}{4}$

b) $E[Y|X=x] = \begin{cases} \frac{1}{2}(1+3), & x=0 \\ \frac{1}{2}(1+2), & x=2 \\ \frac{1}{4}(0+1+2+3), & x=4 \end{cases} = \begin{cases} 2, & x=0 \\ 3/2, & x=2 \\ 3/2, & x=4 \end{cases}$

This is maximized for $x=0$.

c) A denotes event $X^2 \geq Y$

$P(A) = P(\{(2,1),(2,2),(4,0),(4,1),(4,2),(4,3)\}) = \frac{6}{8} = \frac{3}{4}$

$E(XY|A) = \sum_{x,y} xy\, P_{XY|A}(xy|A) = \frac{1/8}{3/4}(2+4+0+4+8+12)$

$= \frac{1}{6}(30) = \boxed{5}$

4.6) X is R.V. with pdf $f_X(\alpha)$ and c.f. $M_X(ju) = E[e^{juX}]$

$Y = aX + b \qquad a, b > 0$

Find pdf f_Y and c.f. $M_Y(ju)$ in terms of f_X, M_X

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(\alpha)\, d\alpha$$

$$F_Y(y) = P(Y \le y) = P(aX + b \le y) = P(aX \le y-b) = P\left(X \le \frac{y-b}{a}\right)$$

$$= F_X\left(\frac{y-b}{a}\right) = \int_{-\infty}^{\frac{y-b}{a}} f_X(\alpha)\, d\alpha$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d}{dy}\int_{-\infty}^{\frac{y-b}{a}} f_X(\alpha)\, d\alpha = \boxed{\frac{1}{a} f_X\left(\frac{y-b}{a}\right)}$$ Formula 3.41

$$M_Y(ju) = E[e^{juY}] = E[e^{ju(aX+b)}] = E[e^{juaX} e^{jub}] = e^{jub} E[e^{juaX}]$$

$$= \underline{e^{jub} M_X(jua)}$$

4.7) X, Y, Z are iid Gaussian with $N(1,1)$

$V = 2X + Y$

$W = 3X - 2Z + 5$

$var(X) = 1 = E[X^2] - (E[X])^2 = E[X^2] - 1$

$E[X^2] = 2$

a) $E[VW] = E[(2X+Y)(3X-2Z+5)]$

$= E[(6X^2 - 4XZ + 10X + 3XY - 2YZ + 5Y)]$

$= 6E[X^2] - 4E[X]E[Z] + 10E[X] + 3E[X]E[Y] - 2E[Y]E[Z] + 5E[Y]$

$= 6(2) - 4 + 10 + 3 - 2 + 5$

$= 12 + 12$

$= 24$

b) The two parameters that completely specify $V + W$ are the mean and variance because they are Gaussian.

$V + W = 2X + Y + 3X - 2Z + 5$

$= 5X + Y - 2Z + 5$

$E[V+W] = E[5X + Y - 2Z + 5] = 5E[X] + E[Y] - 2E[Z] + 5$

$= 5 + 1 - 2 + 5 = 9$

$var(V+W) = var(5X + Y - 2Z + 5) = 5^2 var(X) + var(Y) + 4 var(Z)$

$= 25 + 1 + 4 = 30$

This is completely specified by mean of 9 and variance of 30.

c) Find c.f. of random vector $[V\ W]^t$

C.f. for Gaussian is

$E[V] = 2E[X] + E[Y] = 2 + 1 = 3$

$E[W] = 3E[X] - 2E[Z] + 5 = 6$

$var(V) = 4 var(X) + var(Y) = 5$

$var(W) = 9 var(X) + 4 var(Z) = 13$

$cov(V, W) = E[VW] - E[V]E[W] = 24 - (3)(6) = 6$

This is a cascade vector with $K_{(v,w)} = \begin{bmatrix} 5 & 6 \\ 6 & 13 \end{bmatrix}$ and $\vec{m} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$M(j\vec{u}) = e^{j\vec{u}^t\vec{m} - \vec{u}^t \Lambda \vec{u}/2}$ where $\vec{m} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ and $\Lambda = \begin{bmatrix} 5 & 6 \\ 6 & 13 \end{bmatrix}$

d) The linear estimator $\hat{\underline{V}}(\underline{W})$ of $\underline{V}$, given $\underline{W}$ is of the form $\hat{\underline{V}}(\underline{W}) = A\underline{W}$

$var(w) = E[$

$$R_{\underline{W}} = E[\underline{W}\,\underline{W}^t] = E[\underline{W}^2] = var(w) + (E[\underline{W}])^2 = 13 + 6^2 = 13 + 36 = 49$$

$$A^t = R_{\underline{W}}^{-1}\, E[\underline{W}\,\underline{V}^t] = \frac{1}{49} \cdot 24$$

$$\hat{\underline{V}}(\underline{W}) = \frac{24}{49}\underline{W}$$

e) The optimal (smallest MSE) affine estimator is of form

$$\hat{\underline{V}}(\underline{W}) = a\underline{W} + b$$

$$a = \frac{cov(\underline{V}, \underline{W})}{var(\underline{W})} = \frac{6}{13}$$

$$b = E[\underline{V}] - E[\underline{W}]\,\frac{cov(\underline{V}, \underline{W})}{var(\underline{W})} = 3 - 6\left(\frac{6}{13}\right) = 3 - \frac{36}{13} = \frac{39-36}{13} = \frac{3}{13}$$

$$\hat{\underline{V}}(\underline{W}) = \frac{6}{13}\underline{W} + \frac{3}{13}$$

f) This is the optimal estimator because $\underline{V}$ and $\underline{W}$ are both Gaussian

g) $\underline{X} - \overline{\underline{X}}$, $\underline{Y} - \overline{\underline{Y}}$, $\underline{Z} - \overline{\underline{Z}}$ inputs to black box

Two outputs A and B

$\begin{bmatrix} A \\ B \end{bmatrix}$ You want $\Lambda_{AB} = \begin{bmatrix} 3 & 2 \\ 2 & 5 \end{bmatrix}$

This means you have $var(A) = 3$ and $var(B) = 5$ and $cov(A, B) = 2$.

You have $A = a_1\mathring{\underline{X}} + a_2\mathring{\underline{Y}} + a_3\mathring{\underline{Z}}$ where $\mathring{\underline{X}} = \underline{X} - \overline{\underline{X}}$ $N(0,1)$ $E[\mathring{\underline{X}}] = 0$

$B = b_1\mathring{\underline{X}} + b_2\mathring{\underline{Y}} + b_3\mathring{\underline{Z}}$ $\quad \mathring{\underline{Y}} = \underline{Y} - \overline{\underline{Y}}$ $N(0,1)$ $E[\mathring{\underline{X}}]^2 = 1$

$\mathring{\underline{Z}} = \underline{Z} - \overline{\underline{Z}}$ $N(0,1)$

$$cov(A,B) = 2 = E[AB] - E[A]E[B] = a_1 b_1 E[\mathring{\underline{X}}^2] + a_2 b_2 E[\mathring{\underline{Y}}^2] + a_3 b_3 E[\mathring{\underline{Z}}^2]$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3$$

$var(A) = 3 = a_1^2 + a_2^2 + a_3^2$

$var(B) = 5 = b_1^2 + b_2^2 + b_3^2$

An example thus is $a_1 b_1 = 2$

$a_1 = 1$, $b_1 = 2$

$a_2 = \sqrt{2}$, $b_2 = 0$

$a_3 = 0$, $b_3 = 1$

g cont...)

This would satisfy this

$$A = \mathring{X} + \sqrt{2}\,\mathring{Y}$$
$$B = 2\mathring{X} + \mathring{Z}$$

$cov(A,B) = E[AB] - E[A]E[B] = E[2\mathring{X}^2] = 2E[\mathring{X}^2] = 2$

$var(A) = var(\mathring{X}) + 2\,var(\mathring{Y}) = 3$

$var(B) = 4\,var(\mathring{X}) + var(\mathring{Z}) = 5$

h) $\Lambda_{CD} = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$

This means that C and D are uncorrelated. Thus, using an affine estimator will not be useful at all. If the inputs are Gaussians and the outputs are Gaussians, the C and D are independent and output D gives you no information about output C.

4.19) a) $E[(X_t - X_s)^2] \geq 0$

If $EX_t = EX_0$ for all t

$E(X_t^2) = R_X(t,t) = R_X(0,0)$ for all t

$$E[(X_t - X_s)^2] \geq 0$$

$$E[X_t^2 - 2X_tX_s + X_s^2] \geq 0$$

$$E[X_t^2] - 2E[X_tX_s] + E[X_s^2] \geq 0$$

$$2R_X(0,0) \geq 2E[X_tX_s]$$

$$E[X_tX_s] \leq R_X(0,0)$$

$$R_X(t,s) \leq R_X(0,0)$$

It is also true that

$$E[(X_t + X_s)^2] \geq 0$$

$$E[X_t^2 + 2X_tX_s + X_s^2] \geq 0$$

$$E[X_t^2] + 2E[X_tX_s] + E[X_s^2] \geq 0$$

$$2R_X(0,0) \geq -2E[X_tX_s]$$

$$-R_X(t,s) \leq R_X(0,0)$$

If $R_X(t,s) \leq R_X(0,0)$ and $-R_X(t,s) \leq R_X(0,0)$

then $|R_X(t,s)| \leq R_X(0,0)$

$$K_X(t,s) = R_X(t,s) - E[X_t]E[X_s] = R_X(t,s) - (EX_0)^2$$

$$K_X(0,0) = R_X(0,0) - E[X_0]E[X_0] = R_X(0,0) - (EX_0)^2$$

$$R_X(t,s) = K_X(t,s) + (EX_0)^2$$

$$R_X(0,0) = K_X(0,0) + (EX_0)^2$$

$$R_X(t,s) \leq R_X(0,0)$$

$$K_X(t,s) + (EX_0)^2 \leq K_X(0,0) + (EX_0)^2$$

$$K_X(t,s) \leq K_X(0,0)$$

$$K_X(t,t) = R_X(t,t) - E[X_t]E[X_t] = R_X(0,0) - E[X_0]E[X_0] = K_X(0,0)$$

$$E[((X_t - EX_t) - (X_s - EX_s))^2] \geq 0$$

$$E[(X_t - EX_t)^2 - 2(X_t - EX_t)(X_s - EX_s) + (X_s - EX_s)^2] \geq 0$$

$$K_X(t,t) - 2K_X(t,s) + K_X(s,s) \geq 0$$

$$2K_X(0,0) - 2K_X(t,s) \geq 0$$

$$K_X(t,s) \leq K_X(0,0)$$

$$E[((X_t - EX_t) + (X_s - EX_s))^2] \geq 0$$

$$E[(X_t - EX_t)^2 + 2(X_t - EX_t)(X_s - EX_s) + (X_s - EX_s)^2] \geq 0$$

$$K_X(t,t) + 2K_X(t,s) + K_X(s,s) \geq 0$$

$$2K_X(0,0) + 2K_X(t,s) \geq 0$$

$$-K_X(t,s) \leq K_X(0,0)$$

If $K_X(t,s) \leq K_X(0,0)$ and $-K_X(t,s) \leq K_X(0,0)$ then

$$|K_X(t,s)| \leq K_X(0,0)$$

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b) Autocorrelation and covariance functions are symmetric

$$R_X(t,s) = E[X_t X_s] = E[X_s X_t] = R_X(s,t)$$

$$K_X(t,s) = E[X_t X_s] - E[X_t]E[X_s] =$$

$$E[X_s X_t] - E[X_s]E[X_t] = K_X(s,t)$$

4.22) Θ uniform on $[-\pi, \pi]$

Y has mean m and variance σ^2

Θ and Y are independent

$\{X(t);\ t \in \mathbb{R}\}$ $\quad X(t) = Y\cos(2\pi f_0 t + \Theta)$

Find mean and autocorrelation function.

$$E[X(t)] = E[Y\cos(2\pi f_0 t + \Theta)] = E[Y]\,E[\cos(2\pi f_0 t + \Theta)] \quad (\to 0)$$
$$= 0$$

$$R_X(t,s) = E[X_t X_s] \text{ ; all } t,s \in \mathcal{T}$$

$$\mathrm{var}(Y) = E[Y^2] - (E[Y])^2$$
$$E[Y^2] = \sigma^2 + m^2$$

$$= E[(Y\cos(2\pi f_0 t + \Theta))(Y\cos(2\pi f_0 s + \Theta))]$$
$$= E[Y^2]\,E[\cos(2\pi f t + \Theta)\cos(2\pi f s + \Theta)]$$
$$= (\sigma^2 + m^2)\,E[\cos(2\pi f t + \Theta)\cos(2\pi f s + \Theta)]$$
$$= (\sigma^2 + m^2)\int_{-\pi}^{\pi} \frac{1}{2\pi}\cos(2\pi f t + \Theta)\cos(2\pi f s + \Theta)\,d\Theta$$

$$= \frac{(\sigma^2 + m^2)}{2\pi}\int_{-\pi}^{\pi}\Big[[\cos(2\pi f t)\cos(\Theta) - \sin(2\pi f t)\sin(\Theta)][\cos(2\pi f s)\cos(\Theta) - \sin(2\pi f s)\sin(\Theta)]\Big]\,d\Theta$$

$$= \frac{(\sigma^2 + m^2)}{2\pi}\int_{-\pi}^{\pi}\Big[\cos(2\pi f t)\cos(2\pi f s)\cos^2(\Theta) - \sin(2\pi f t)\cos(2\pi f s)\sin\Theta\cos(\Theta) - \sin(2\pi f s)\cos(2\pi f t)\sin(\Theta)\cos(\Theta)$$
$$+ \sin(2\pi f t)\sin(2\pi f s)\sin^2(\Theta)\Big]\,d\Theta$$

$$\left\{\int_{-\pi}^{\pi}(\sin\Theta)(\cos\Theta)\,d\Theta = \tfrac{1}{2}\sin^2\Theta\Big]_{-\pi}^{\pi} = 0\right\}$$

$$= \frac{(\sigma^2 + m^2)}{2\pi}\int_{-\pi}^{\pi}\big[\cos(2\pi f t)\cos(2\pi f s)\cos^2\Theta + \sin(2\pi f t)\sin(2\pi f s)\sin^2(\Theta)\big]\,d\Theta$$

$$= \frac{(\sigma^2 + m^2)}{2\pi}(\pi)\big[\cos(2\pi f t)\cos(2\pi f s) + \sin(2\pi f t)\sin(2\pi f s)\big]$$

$$\int_{-\pi}^{\pi}\cos^2\Theta = \left[\tfrac{1}{2}\Theta + \tfrac{1}{4}\sin 2\Theta\right]_{-\pi}^{\pi} = \frac{2\pi}{2} = \pi$$
$$\int_{-\pi}^{\pi}\sin^2\Theta = \left[\tfrac{1}{2}\Theta - \tfrac{1}{4}\sin 2\Theta\right]_{-\pi}^{\pi} = \frac{2\pi}{2} = \pi$$

$$= \frac{\sigma^2 + m^2}{2}\big[\cos(2\pi f t - 2\pi f s)\big]$$
$$= \frac{\sigma^2 + m^2}{2}\cos(2\pi f(t-s))$$

4.22 cont...)

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X(t)\,dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T Y \cos(2\pi f_0 t + \theta)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \left[\frac{Y \sin(2\pi f_0 t + \theta)}{2\pi f_0} \right]_0^T$$

$$= \lim_{T\to\infty} \frac{1}{T} \left(\frac{Y}{2\pi f_0} \right) \left[\sin(2\pi f_0 T + \theta) - \sin(\theta) \right]$$

$$\left| \frac{Y}{2\pi f_0} \left[\sin(2\pi f_0 T + \theta) - \sin(\theta) \right] \right| \le \left| \frac{2Y}{2\pi f_0} \right|$$

$$\left| \lim_{T\to\infty} \frac{1}{T} \int_0^T X(t)\,dt \right| \le \lim_{T\to\infty} \frac{1}{T} \left| \frac{2Y}{2\pi f_0} \right| = 0 \quad \text{because } \left| \frac{2Y}{2\pi f_0} \right| \text{ is finite}$$

$$\therefore \lim_{T\to\infty} \frac{1}{T} \int_0^T X(t)\,dt = 0.$$

4.26) $\hat{X}_n = a X_{n-1} + b$

$$\epsilon \triangleq E[(X_n - \hat{X}_n)^2]$$

$$= E[X_n^2] - 2E X_n \hat{X}_n + E[\hat{X}_n^2]$$

$$= R_X(n,n) - 2aE[X_n X_{n-1}] - 2bE[X_n] + a^2 E[X_{n-1}^2] + 2abE[X_{n-1}] + b^2$$

$$= R_X(n,n) - 2aR_X(n,n-1) - 2bm_n + a^2 R_X(n-1,n-1) + 2abm_{n-1} + b^2$$

We want the derivative to be zero:

$$0 = \frac{d\epsilon}{db} = -2m_n + 2am_{n-1} + 2b$$

$$-m_n + am_{n-1} + b = 0$$

$$b = m_n - am_{n-1}$$

Notice $\epsilon = E[((X_n - m_n) - a(X_{n-1} - m_{n-1}))^2]$

$= K_X(n,n) - 2a K_X(n-1,n-1)$

$+ a^2 K_X(n-1,n-1)$

$$\epsilon = R_X(n,n) - 2aR_X(n,n-1) - 2(m_n - am_{n-1})m_n + a^2 R_X(n-1,n-1) + 2a(m_n - am_{n-1})m_{n-1}$$
$$+ (m_n - am_{n-1})^2$$

$$= R_X(n,n) + a^2 R_X(n-1,n-1) - 2aR_X(n,n-1)$$
$$-2m_n^2 + 2am_n m_{n-1} + 2am_n m_{n-1} - 2a^2 m_{n-1}^2 + m_n^2 - 2am_n m_{n-1} + a^2 m_{n-1}^2$$

$$= K_X(n,n) + a^2 K_X(n-1,n-1) - 2a K_X(n,n-1)$$

$$0 = \frac{d\epsilon}{da} = -2K_X(n,n-1) + 2a K_X(n-1,n-1)$$

$$a = \frac{K_X(n,n-1)}{K_X(n-1,n-1)}$$

$$b = m_n - \frac{K_X(n,n-1)}{K_X(n-1,n-1)} m_{n-1}$$

4.26 cont...) $\hat{X}_n(X_{n-1}, X_{n-m}) = a_1 X_{n-1} + a_m X_{n-m} + b$.

$$\epsilon \triangleq E[(X_n - \hat{X}_n)^2]$$
$$= E[X_n^2] - 2E[X_n \hat{X}_n] + E[\hat{X}_n^2]$$
$$= R_X(n,n) - 2E[X_n(a_1 X_{n-1} + a_m X_{n-m} + b)] + E[(a_1 X_{n-1} + a_m X_{n-m} + b)^2]$$
$$= R_X(n,n) - 2a_1 R_X(n,n-1) - 2a_m R_X(n,n-m) - 2b m_n$$
$$+ a_1^2 R_X(n-1,n-1) + a_m^2 R_X(n-m,n-m) + b^2$$
$$+ 2a_1 a_m R_X(n-1,n-m) + 2b(a_1 m_{n-1} + a_m m_{n-m})$$

$$0 = \frac{d\epsilon}{db} = -2m_n + 2b + 2(a_1 m_{n-1} + a_m m_{n-m})$$
$$0 = -m_n + b + a_1 m_{n-1} + a_m m_{n-m}$$
$$b = m_n - a_1 m_{n-1} - a_m m_{n-m}$$

$$\epsilon = R_X(n,n) - 2a_1 R_X(n,n-1) - 2a_m R_X(n,n-m) - 2(m_n - a_1 m_{n-1} - a_m m_{n-m}) m_n$$
$$+ a_1^2 R_X(n-1,n-1) + a_m^2 R_X(n-m,n-m) + (m_n - a_1 m_{n-1} - a_m m_{n-m})^2$$
$$+ 2a_1 a_m R_X(n-1,n-m) + 2(m_n - a_1 m_{n-1} - a_m m_{n-m})(a_1 m_{n-1} + a_m m_{n-m})$$
$$= K_X(n,n) - 2a_1 K_X(n,n-1) - 2a_m K_X(n,n-m) + a_1^2 K_X(n-1,n-1)$$
$$+ a_m^2 K_X(n-m,n-m) + 2a_1 a_m K_X(n-1,n-m)$$

$$0 = \frac{\partial \epsilon}{\partial a_1} = -2K_X(n,n-1) + 2a_1 K_X(n-1,n-1) + 2a_m K_X(n-1,n-m)$$

$$0 = \frac{\partial \epsilon}{\partial a_m} = -2K_X(n,n-m) + 2a_m K_X(n-m,n-m) + 2a_1 K_X(n-1,n-m)$$

$$K_X(n,n-1) = a_1 K_X(n-1,n-1) + a_m K_X(n-1,n-m)$$
$$K_X(n,n-m) = a_1 K_x(n-1,n-m) + a_m K_X(n-m,n-m)$$
$$a_m = \frac{K_X(n,n-m) - a_1 K_X(n-1,n-m)}{K_X(n-m,n-m)}$$

$$a_1 = \frac{K_X(n-1,n-m) K_X(n,n-m) - K_X(n,n-1) K_X(n-m,n-m)}{K_X(n-1,n-m)^2 - K_X(n-1,n-1) K_X(n-m,n-m)}$$

$$a_m = \frac{K_X(n-1,n-m) K_X(n,n-1) - K_X(n,n-m) K_X(n-1,n-1)}{K_X(n-1,n-m)^2 - K_X(n-1,n-1) K_X(n-m,n-m)}$$

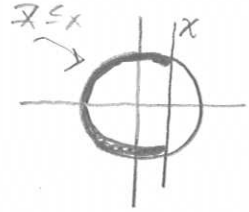
4.26 cont...)

$a_m = 0$ when $K_X(n-1, n-m)\,K_X(n, n-1) - K_X(n, n-m)\,K_X(n-1, n-1) = 0$

This is true if the process is stationary or it's first-order Markov.

4.29)

a) $F_{X(0)}(x) = \Pr(X(0) \le x) = \Pr(\cos\phi \le x)$



For $x < -1$, $F_{X(0)}(x) = 0$

For $x > 1$, $F_{X(0)}(x) = 1$

$$F_{X(0)}(x) = \Pr\left(\phi \in [-\pi, -\cos^{-1}(x)] \cup [\cos^{-1}(x), \pi]\right)$$

$$= 1 - \Pr\left(\phi \in (-\cos^{-1}(x), \cos^{-1}(x))\right)$$

$$= 1 - \int_{-\cos^{-1}(x)}^{\cos^{-1}(x)} \frac{d\phi}{2\pi}$$

$$= 1 - \frac{\cos^{-1}(x)}{\pi}$$

$$F_{X(0)}(x) = \begin{cases} 0, & x < -1 \\ 1 - \dfrac{\cos^{-1}(x)}{\pi}, & -1 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

b) $EX(t) = E[\cos(t+\phi)] = \int_{-\pi}^{\pi} \cos(t+\phi)\, d\phi \frac{1}{2\pi} = \underline{0}$

c) $K_X(t,s) = E[X(t)X(s)] = E[\cos(t+\theta)\cos(s+\theta)]$

$$= \int_{-\pi}^{\pi} \cos(t+\phi)\cos(s+\phi)\frac{d\phi}{2\pi}$$

$$\cancel{= \int_{-\pi}^{\pi} \left(\frac{1}{2}\cos(t+s+2\phi) + \frac{1}{2}\cos(t+s)\right)\frac{d\phi}{2\pi}}$$

$$\cancel{= \underline{\frac{1}{2}\cos(t+s)}}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} [\cos(t)\cos\phi - \sin(t)\sin\phi][\cos(s)\cos\phi - \sin(s)\sin\phi]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \cos(s)\cos(t)\cos^2\phi - \sin(t)\cos(s)\sin\phi\cos\phi - \sin(s)\cos(t)\sin\phi\cos\phi + \sin(s)\sin(t)\sin^2\phi$$

$$= \frac{1}{2\pi}\left(\pi\left(\cos(s)\cos(t) - \sin(s)\sin(t)\right)\right) = \frac{1}{2}\cos(s-t)$$

4.30) a) $E[Y_n] = E[X_n + W_n] = EX_n + EW_n = m$

$$R_Y(n,k) = E[(X_n + W_n)(X_k + W_k)]$$

$$= E[X_n X_k] + \cancel{E[X_n W_k]}^{0} + \cancel{E[W_n X_k]}^{0} + E[W_n W_k]$$

$$= R_X(n,k) + R_W(n,k)$$

$$= R_X(n,k) + \sigma_W^2 \delta_{n-k}$$

$$K_Y(n,k) = R_Y(n,k) - EY_n EY_k = R_X(n,k) + \sigma_W^2 \delta_{n-k} - m^2$$

$$= K_X(n,k) + \sigma_W^2 \delta_{n-k}$$

b) $R_{XY}(k,j) = E[X_k Y_j]$

$$= E[X_k (X_j + W_j)]$$

$$= E[X_k X_j] + E[X_k W_j]$$

$$= R_X(k,j)$$

c) This is the same as Problem 4.26

$$\hat{X}_n = a \underset{\substack{\uparrow \\ X_{n-1} \text{ in } 4.26}}{Y_n} + b$$

$$a = \underbrace{\frac{K_X(n,n-1)}{K_X(n-1,n-1)}}_{4.26} = \frac{E[X_n Y_n] - E[X_n]\overset{0}{E[Y_n]}}{E[Y_n Y_n] - E[Y_n]\underset{0}{E[Y_n]}} = \frac{E[X_n Y_n]}{E[Y_n^2]} = \boxed{\frac{R_X(n,n)}{R_X(n,n) + \sigma_W^2}}$$

$$b = \underbrace{m_n - \frac{K_X(n,n-1)}{K_X(n-1,n-1)}}_{4.26} \overset{m_{Y_{n-1}}}{m_{n-1}} = m_{X_n} - \frac{R_X(n,n)}{R_X(n,n) + \sigma_W^2} m_{Y_n} = \boxed{m - \frac{R_X(n,n)}{R_X(n,n) + \sigma_W^2} m}$$

4.30 d) Using 4.26

$$4.26: \hat{X}_n(X_{n-1}, X_{n-m}) = a_1 X_{n-1} + a_m X_{n-m} + b$$

$$4.30: \hat{X}_n(Y_n, Y_{n-1}) = a_1 Y_n + a_2 Y_{n-1} + b$$

4.26:

$$a_1 = \frac{K_X(n-1,n-m)\,K_X(n,n-m) - K_X(n,n-1)\,K_X(n-m,n-m)}{K_X(n-1,n-m)^2 - K_X(n-1,n-1)\,K_X(n-m,n-m)}$$

4.30:

$$a_1 = \frac{K_Y(n,n-1)\,K_{XY}(n,n-1) - K_{XY}(n,n)\,K_Y(n-1,n-1)}{K_Y(n,n-1)^2 - K_Y(n,n)\,K_Y(n-1,n-1)} = \frac{(R_X(n,n-1)-m^2)R_X(n,n-1) - R_X(n,n)[R_X(n,n)+\sigma_w^2]}{(R_X(n,n-1)-m^2)^2 - (R_X(n,n)+\sigma_w^2)(R_X(n-1,n-1)+\sigma_w^2)}$$

4.26:

$$a_m = \frac{K_X(n-1,n-m)\,K_X(n,n-1) - K_X(n,n-m)\,K_X(n-1,n-1)}{K_X(n-1,n-m)^2 - K_X(n-1,n-1)\,K_X(n-m,n-m)}$$

4.30:

$$a_2 = \frac{K_Y(n,n-1)\,K_{XY}(n,n) - K_{XY}(n,n-1)\,K_Y(n,n)}{K_Y(n,n-1)^2 - K_Y(n,n)\,K_Y(n-1,n-1)} = \frac{(R_X(n,n-1)-m^2)R_X(n,n) - R_X(n,n-1)[R_X(n,n)+\sigma_w^2]}{(R_X(n,n-1)-m^2)^2 - (R_X(n,n)+\sigma_w^2)(R_X(n-1,n-1)+\sigma_w^2)}$$

$$4.26: b = m_n - a_1 m_{n-1} - a_m m_{n-m}$$

$$4.30: b = m_{X_n} - a_1 m_{Y_n} - a_2 m_{Y_{n-1}}$$

$$\boxed{b = m(1 - a_1 - a_2)}$$

4.31)

$$\mathbb{Y}_1 = \mathbb{W}_1$$
$$\mathbb{Y}_2 = \mathbb{W}_2$$
$$\mathbb{Y}_3 = \mathbb{W}_1 \oplus \mathbb{W}_2$$

$$p_{\mathbb{Y}_1}(x) = p_{\mathbb{X}_2}(x) = p_{\mathbb{Y}_3}(x) = \frac{1}{2}\ ;\ x = 0, 1$$

$p_{\mathbb{X}_i}(x) = \frac{1}{2}\ ;\ x = 0, 1$, identically distributed

$p_{\mathbb{X}_i, \mathbb{X}_j}(a, b)$ has one of the following three forms, $p_{\mathbb{Y}_1, \mathbb{Y}_2}$, $p_{\mathbb{Y}_1, \mathbb{Y}_3}$, $p_{\mathbb{Y}_2, \mathbb{Y}_3}$

$p_{\mathbb{Y}_1, \mathbb{Y}_2} = p_{\mathbb{Y}_1} p_{\mathbb{Y}_2}$ because $\mathbb{W}_1$ and $\mathbb{W}_2$ are independent.

$$p_{\mathbb{Y}_1, \mathbb{Y}_3}(a, b) = \Pr(\mathbb{W}_1(n) = a,\ W_1(n) \oplus W_2(n) = b)$$
$$= \underbrace{\Pr(W_1(n) \oplus W_2(n) = b \mid \mathbb{W}_1(n) = a)}_{\Pr(W_2(n) = a \oplus b) = 1/2}\ \underbrace{\Pr(W_1(n) = a)}_{1/2}$$
$$= \frac{1}{4} \quad \forall a, b$$

$$p_{\mathbb{Y}_2, \mathbb{Y}_3} = p_{\mathbb{Y}_2} p_{\mathbb{Y}_3}$$

The process is not iid. If $\mathbb{X}_0$ and $\mathbb{X}_1$ are known, then $\mathbb{X}_2 = \mathbb{X}_0 \oplus \mathbb{X}_1$.

$$p_{\mathbb{X}_0, \mathbb{X}_1, \mathbb{X}_2} \neq \prod_{i=0}^{2} p_{\mathbb{X}_i}$$