

ECEn 670

Homework Problem Set 5

Due at beginning of class, Thursday, November 5, 2009

Problems are from *An Introduction to Statistical Signal Processing* by Gray and Davisson unless otherwise specified.

1. 4.20
2. 4.36
3. 4.38
4. 4.39
5. 4.51
6. 4.52

4.20

Assume that $a = [E[X^2]]^{1/2} > 0$
 $b = [E[Y^2]]^{1/2} > 0$

$$0 \leq E\left[\left(\frac{X}{a} \pm \frac{Y}{b}\right)^2\right] = \frac{E[X^2]}{a^2} + \frac{E[Y^2]}{b^2} \pm \frac{2E[XY]}{ab}$$

$$= 2 \pm \frac{2E[XY]}{ab}$$

↓

$$0 \leq 1 \pm \frac{E[XY]}{(E[X^2])^{1/2}(E[Y^2])^{1/2}}$$

This proves that

$$|E[XY]| \leq [E[X^2]]^{1/2} [E[Y^2]]^{1/2}$$

for $a, b > 0$.

For $a = 0$, we have $E[X^2] = 0$

Suppose X can take a value $c \neq 0$ with probability $p > 0$. This implies that $E[X^2] = \sum x^2 p(x) > 0$. If we say that $E[X^2] = 0$, then $P(X=0) = 1$. This implies that $E[XY] = 0$, and so $|E[XY]| \leq E[X^2]^{1/2} E[Y^2]^{1/2}$ in this case. This also applies to $b = 0$.

Homework 5 Solution: 4.36
ECEn 670, Fall 2009

4.36 The purpose of this problem is to demonstrate the relationships among the four forms of convergence that we have presented. In each case $([0, 1], \mathcal{B}([0, 1]), P)$ is the underlying probability space, with probability measure described by the uniform pdf. For each of the following sequences of random variables, determine the pmf of $\{Y_n\}$, the senses in which the sequences converge, and the random variable and pmf to which the sequences converge.

$$a) Y_n(\omega) = \begin{cases} 1 & \text{if } n \text{ is odd and } \omega < 1/2 \text{ or } n \text{ is even and } \omega > 1/2 \\ 0 & \text{otherwise} \end{cases}$$

What we get out of this sequence is either 10101010101010... or 010101010101... which are determined by the choice of ω at the beginning of the experiment. We see that the pmf for this can be written as

$$p_{Y_n}(y) = \begin{cases} \frac{1}{2}, & y = 1 \\ \frac{1}{2}, & y = 0 \end{cases}$$

We should note that there is no dependence on the position of each random variable because there is no dependence on n . Thus, this sequence will converge in distribution to a random variable with the same pmf. However, this also means that we may have a hard time getting any other type of convergence. Let us suppose that this then converges to Y_1 in the sequence.

We can compute the mean square convergence criteria:

$$\lim_{n \rightarrow \infty} E[(Y_n - Y)^2] = \lim_{n \rightarrow \infty} E[(Y_n - Y_1)^2] = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

This obviously does not converge in mean square then. We can see if it converges in probability. This would imply that $\lim_{n \rightarrow \infty} \Pr(|Y_n - Y| > \epsilon) = 0$ for any value of $\epsilon > 0$. Well, we see that we can't converge to any particular value so again we can try convergence to Y_1 . In this case we see that if $0 < \epsilon < 1$

$$\lim_{n \rightarrow \infty} \Pr(|Y_n - Y_1| > \epsilon) = \Pr(|Y_n - Y_1| > \epsilon) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

Thus, this is not convergent in probability. This means it is definitely not convergent with probability one. It is not convergent in mean square. It is convergent only in distribution.

$$b) Y_n(\omega) = \begin{cases} 1 & \text{if } \omega < 1/n \\ 0 & \text{otherwise} \end{cases}$$

I think of this sequence as a whole lot of 1's, then when the threshold is crossed, that then go to 0's. If we look at the definition of convergence with probability 1, then we can actually compute the criteria for this convergence, $\lim_{n \rightarrow \infty} Y_n(\omega) = Y(\omega)$.

$$\lim_{n \rightarrow \infty} Y_n = \begin{cases} 1 & \text{if } \omega < \lim_{n \rightarrow \infty} \frac{1}{n} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } \omega = 0 \\ 0 & \text{otherwise} \end{cases}$$

We note that $\Pr(\omega = 0) = 0$. Thus, as the limit increases to infinity, we see that $\Pr(Y_n = 0) = 1$. This implies that $Y_n(\omega)$ converges to $Y(\omega) = 0$ with probability one. This then implies that it also converges in probability and in distribution. How about mean square? Well, it is useful to actually look at the pmf to which this sequence converges. The pmf can be defined as follows:

$$p_{Y_n}(y) = \begin{cases} \frac{1}{n}, & y = 1 \\ 1 - \frac{1}{n}, & y = 0 \end{cases}$$

The limiting pmf is thus:

$$p_{Y_n}(y) = \begin{cases} 0, & y = 1 \\ 1, & y = 0 \end{cases}$$

This is useful so we can compute the expectation of Y_n . So, we now look at mean square convergence.

$$\lim_{n \rightarrow \infty} E[(Y_n - Y)^2] = \lim_{n \rightarrow \infty} E[(Y_n - 0)^2] = \lim_{n \rightarrow \infty} E[Y_n^2] = \lim_{n \rightarrow \infty} 1 \cdot \frac{1}{n} + 0 \cdot \left(1 - \frac{1}{n}\right) = 0.$$

Thus, this sequence definitely converges in the mean square sense.

$$c) Y_n(\omega) = \begin{cases} n & \text{if } \omega < 1/n \\ 0 & \text{otherwise} \end{cases}$$

This is in some ways just a variation of part b, but with a twist that we see that n is going to increase to infinity even though the probability of choosing n decreases as well. Who wins? Let's analyze it.

$$\lim_{n \rightarrow \infty} Y_n = \begin{cases} n & \text{if } \omega < \lim_{n \rightarrow \infty} \frac{1}{n} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \infty & \text{if } \omega = 0 \\ 0 & \text{otherwise} \end{cases}$$

We note that $\Pr(\omega = 0) = 0$. Thus, as the limit increases to infinity, we see that $\Pr(Y_n = 0) = 1$. Thus, this definitely converges to $Y_n(\omega) = 0$ with probability one. This in turn implies that this converges in probability and in distribution. Let's now examine mean square.

The pmf can be defined as follows:

$$p_{Y_n}(y) = \begin{cases} \frac{1}{n}, & y = n \\ 1 - \frac{1}{n}, & y = 0 \end{cases}$$

This is useful so we can compute the expectation of Y_n . So, we now look at mean square convergence.

$$\lim_{n \rightarrow \infty} E[(Y_n - Y)^2] = \lim_{n \rightarrow \infty} E[(Y_n - 0)^2] = \lim_{n \rightarrow \infty} E[Y_n^2] = \lim_{n \rightarrow \infty} n^2 \cdot \frac{1}{n} + 0 \cdot \left(1 - \frac{1}{n}\right) = \infty.$$

We see that this diverges and thus this definitely does not converge in mean square. The limiting pmf is thus:

$$p_{Y_n}(y) = \begin{cases} 0, & y = \infty \\ 1, & y = 0 \end{cases}$$

which is not particularly useful.

d) Divide $[0,1]$ into a sequence of intervals $\{F_n\} = \{[0, 1], [0, \frac{1}{2}), [\frac{1}{2}, 1], [0, \frac{1}{3}), [\frac{1}{3}, \frac{2}{3}), [\frac{2}{3}, 1], [0, \frac{1}{4}), \dots\}$
Let

$$Y_n(\omega) = \begin{cases} 1 & \text{if } \omega \in F_n \\ 0 & \text{otherwise} \end{cases}$$

Here we see that we will definitely start with a 1, then we will get a 1 in the next 2, then a 1 among the next 3, then a 1 among the next 4, and so on. Thus, the proportion of zeros to ones will definitely increase as the sequence progresses. However, you will always get a one later for any value of ω . Thus we cannot say that $\lim_{n \rightarrow \infty} Y_n(\omega) = 0$ for any ω . This means that this sequence does not converge with probability one because each and every ω does not converge.

However, if we look at mean square convergence we can see that maybe there is something we can do there:

$$\lim_{n \rightarrow \infty} E[(Y_n - Y)^2] = \lim_{n \rightarrow \infty} 1 \cdot \Pr(\omega \in F_n) = \lim_{n \rightarrow \infty} (\text{length of } F_n) = 0$$

Thus, this does converge in mean square. This means that it is also convergent in probability and in distribution.

$$e) Y_n(\omega) = \begin{cases} 1 & \text{if } \omega < 1/2 + 1/n \\ 0 & \text{otherwise} \end{cases}$$

This is very similar to part b but we are converging to another random variable:

$$Y(\omega) = \lim_{n \rightarrow \infty} Y_n(\omega) = \begin{cases} 1 & \text{if } \omega < \lim_{n \rightarrow \infty} \frac{1}{2} + \frac{1}{n} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } \omega < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Because for every instance of ω $Y_n(\omega)$ converges to $Y(\omega)$, this converges with probability one. This also implies a pmf of:

$$p_{Y_n}(y) = \begin{cases} \frac{1}{2} + \frac{1}{n}, & y = 1 \\ \frac{1}{2} - \frac{1}{n}, & y = 0 \end{cases}$$

which in the limit becomes

$$p_{Y_n}(y) = \begin{cases} \frac{1}{2}, & y = 1 \\ \frac{1}{2}, & y = 0 \end{cases}$$

We can also test mean square convergence

$$\lim_{n \rightarrow \infty} \text{E} \left[(Y_n - Y)^2 \right] = \text{E} \left[(Y - Y)^2 \right] = 0.$$

This means that it also converges in probability and distribution.

- 4.38) $\{\mathbf{X}_n\}$ is iid zero-mean Gaussian with $R_{\mathbf{X}}(0) = \sigma^2$
 $\{U_n\}$ is iid process with $\Pr(U_n=1) = \Pr(U_n=-1) = \frac{1}{2}$.
 \mathbf{X}_n and U_n mutually independent
 $\mathbf{Y}_n = U_n \mathbf{X}_n$.

By independence

$$\begin{aligned} \text{a) } R_{\mathbf{Y}}(k, j) &= E[U_k \mathbf{X}_k U_j \mathbf{X}_j] = E[U_k U_j] E[\mathbf{X}_k \mathbf{X}_j] \\ &= \delta_{k-j} R_{\mathbf{X}}(k-j) = \begin{cases} \sigma^2, & k=j \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

$$\begin{aligned} \text{b) } M_{\mathbf{Y}_n}(ju) &= E[e^{ju \mathbf{Y}_n}] = E[e^{ju U_n \mathbf{X}_n}] \\ &= \sum_k P_{U_n}(k) \int dx f_{\mathbf{X}}(x) e^{ju k U_n x} \\ &= \frac{1}{2} \int dx f_{\mathbf{X}}(x) e^{ju x} + \frac{1}{2} \int dx f_{\mathbf{X}}(x) e^{-ju x} \\ &= \frac{1}{2} M_{\mathbf{X}}(ju) + \frac{1}{2} M_{\mathbf{X}}(-ju) \end{aligned}$$

Because $M_{\mathbf{X}}(ju) = M_{\mathbf{X}}(-ju)$

$$= M_{\mathbf{X}}(ju)$$

$$\therefore M_{\mathbf{Y}_n}(ju) = M_{\mathbf{X}}(ju)$$

c) \mathbf{Y}_n is iid:

$$\begin{aligned} \Pr(\mathbf{Y}_i \in F_i; i=1, \dots, n) &= \Pr(U_i \mathbf{X}_i \in F_i; i=1, \dots, n) \\ &= \prod_{i=1}^n \Pr(U_i \mathbf{X}_i \in F_i) \end{aligned}$$

since U_i and \mathbf{X}_i are iid.

$$\text{d) } E[\mathbf{Y}_i] = E[U_i \mathbf{X}_i] = E[U_i] E[\mathbf{X}_i] = 0$$

By the mean ergodic theorem, $S_n = n^{-1} \sum_{i=0}^{n-1} \mathbf{Y}_i$ converges to 0 in mean square.

4.39) $\{X_n\}$ is iid, zero-mean Gaussian with $R_X(0) = \sigma^2$

$\{U_n\}$ is iid, binary with $Pr(U_n=1) = 1-\epsilon$ and $Pr(U_n=0) = \epsilon$

$\{X_n\}$ and $\{U_n\}$ are mutually independent.

$$V_n = X_n U_n$$

a) $E[V_n] = E[X_n U_n] = E[X_n] E[U_n] = 0$

$$M_{V_n}(ju) = E[e^{juV_n}] = E[e^{juX_n U_n}]$$

$$= \sum_k P_{U_n}(k) \int dx f_{X_n}(x) e^{ju x_n u_n}$$

$$= (1-\epsilon) \int dx f_{X_n}(x) e^{ju x_n} + \epsilon \int dx f_{X_n}(x)$$

$$= (1-\epsilon) M_X(ju) + \epsilon$$

$$= (1-\epsilon) e^{-\frac{\sigma^2 u^2}{2}} + \epsilon$$

b) $E[(X_n - V_n)^2] = E[(X_n - X_n U_n)^2] = E[X_n^2 - 2X_n^2 U_n + X_n^2 U_n^2]$

$$= E[X_n^2] - 2E[X_n^2]E[U_n] + E[X_n^2]E[U_n^2]$$

$$= \sigma^2 - 2\sigma^2(1-\epsilon) + \sigma^2(1-\epsilon)$$

$$= \sigma^2(1 - (1-\epsilon))$$

$$= \sigma^2 \epsilon$$

c) $Pr(X_n \neq V_n) = Pr(X_n \neq X_n U_n) = Pr(U_n \neq 1) = \epsilon$

d) cov of V_n .

$$E[V_n] = 0$$

$$K_V(k, j) = E[V_k V_j] = E[X_k U_k X_j U_j]$$

$$= E[X_k X_j] E[U_k U_j] = (\sigma^2 \delta_{k-j}) ((1-\epsilon) \delta_{k-j} + (1-\epsilon)^2 (1 - \delta_{k-j}))$$

$$= \sigma^2 (1-\epsilon) \delta_{k-j}$$

e) Since X_n and U_n are iid, V_n is also i.i.d. By the mean ergodic theorem

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} X_i = E[X_0] = 0$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} U_i = E[U_0] = 1-\epsilon$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} V_i = E[V_0] = 0$$

$$\left. \begin{array}{l} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} X_i = 0 \\ \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} U_i = 1-\epsilon \end{array} \right\} \left(\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} X_i \right) \left(\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} U_i \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} V_i$$

$$0 \quad (1-\epsilon) = 0$$

Yes, this is true

4.51) $\{X_n\}$ is binary iid $P_X(\pm 1) = 0.5$.

$$W_n = X_n + X_{n-1}$$

$$E[W_n] = E[X_n] + E[X_{n-1}] = 0$$

$$\begin{aligned} E[W_n^2] &= E[(X_n + X_{n-1})^2] = E[X_n^2] + 2E[X_n]E[X_{n-1}] + E[X_{n-1}^2] \\ &= (1) + 2(0) + (1) = 2 \end{aligned}$$

$$\text{var}(W_n) = E[W_n^2] - (E[W_n])^2 = 2 - 0 = 2$$

$$\text{cov}(W_j, W_k) = E[W_j W_k]$$

$$= E[(X_j + X_{j-1})(X_k + X_{k-1})]$$

$$= E[X_j X_k + X_{j-1} X_k + X_j X_{k-1} + X_{j-1} X_{k-1}]$$

$$= \begin{cases} 2, & \text{if } j=k \\ 1, & \text{if } |j-k|=1 \\ 0, & \text{otherwise} \end{cases}$$

$\{W_n\}$ is weakly stationary, asymptotically uncorrelated with

$E W_n = 0$ and $\sigma_{W_n}^2 = 2$. By the mean ergodic theorem

$$\text{l.i.m.}_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} W_i = \bar{W} = E W_n.$$

Thus, the sample average also converges in probability, a weak law of large numbers

Use MATLAB

$$4.52) S_{100} = \text{mean}(\text{rand}(100,1)) = 0.5469$$

$$S_{1000} = \text{mean}(\text{rand}(1000,1)) = 0.4917$$

$$S_{10000} = \text{mean}(\text{rand}(10000,1)) = 0.4995$$

To determine if it is random is an active area of research. I'm sure you have multiple solutions.