ECEn 670

Homework Problem Set 6

Due at beginning of class, Thursday, November 19, 2009

Problems are from *An Introduction to Statistical Signal Processing* by Gray and Davisson unless otherwise specified.

1. 5.1
2. 5.7
3. 5.13
4. 5.14
5. 5.15
6. 5.29
7. 5.30

5.1) X_n : iid Gaussian with mean m and variance σ^2.

h : δ-response, $h_0 = 1$, $h_1 = r$, $h_k = 0$ for other k.

$W_n = X_n + r X_{n-1}$

Two-sided:

$$EW_n = E[X_n + rX_{n-1}] = EX_n + rEX_{n-1}$$
$$= (1+r)m$$

$$R_W(k,j) = E[(X_k + rX_{k-1})(X_j + rX_{j-1})]$$
$$= E[X_k X_j] + rE[X_k X_{j-1}] + rE[X_{k-1}X_j] + r^2 E[X_{k-1}X_{j-1}]$$
$$= \begin{cases} (1+r)^2 m^2 + (1+r^2)\sigma^2 & |k-j| = 0 \\ (1+r)^2 m^2 + r^2\sigma^2 & |k-j| = 1 \\ (1+r)^2 m^2 & |k-j| > 1 \end{cases}$$

This process, $\{W_n\}$, is wide-sense stationary. Because W_n is the sum of Gaussians, it too is Gaussian. Because $\{W_n\}$ is wide-sense stationary and Gaussian, it is strictly stationary.

One-sided:

$$E(W_n) = \begin{cases} E(X_0) = m, & n = 0 \\ E(X_n + rX_{n-1}) = (1+r)m, & n > 1 \end{cases}$$

$$R_W(k,j) = \begin{cases} E[X_0^2] = \sigma^2 + m^2 & , k = 0, j = 0 \\ E[X_0(X_1 + rX_0)] = m^2 + r(\sigma^2 + m^2) & , k = 0, j = 1 \text{ or } k = 1, j = 0 \\ E[X_0(X_n + rX_{n-1})] = (1+r)m^2 & k = 0, j > 1, \text{ or } k > 1, j = 0 \\ \text{Same as two sided} & k > 1, j > 1 \end{cases}$$

$$\lim_{n\to\infty} EW_n = (1+r)m$$

$$\lim_{n\to\infty} R_W(n, n+k) = \begin{cases} (1+r)^2 m^2 + (1+r^2)\sigma^2 & , k = 0 \\ (1+r)^2 m^2 + r^2\sigma^2 & , |k| = 1 \\ (1+r)^2 m^2 & , |k| > 1 \end{cases}$$

5.7) $\{X_n\}$ is iid Gaussian with zero mean and $R_X(0) = \sigma^2$.

$\{U_n\}$ iid binary, independent of X, with $Pr(U_n = 1) = Pr(U_n = -1) = 1/2$.

$Z_n = X_n U_n$

$Y_n = U_n + X_n$

$W_n = U_0 + X_n$, all n.

Mean $E[Z_n] = EX_n \, EU_n = 0$

Covariance $K_Z(k,j) = \begin{cases} E[(Z_k - E(Z_k))^2] = E[X_n^2]\,E[U_n^2] = \sigma^2(1) = \sigma^2, & k=j \\ 0, & k \neq j \end{cases}$

PSD $S_Z(f) = \sigma^2$ for all f

Mean $E[Y_n] = E[U_n] + E[X_n] = 0.$

Covariance $K_Y(k,j) = \begin{cases} E[U_k^2] + E[X_k^2] + 2E[U_k X_k] = \sigma^2 + 1 + 0 = \sigma^2 + 1, & k=j \\ 0, & k \neq j \end{cases}$

PSD $S_Y(f) = 1 + \sigma^2$ for all f.

Mean $E[W_n] = E[U_0 + X_n] = E[U_0] + E[X_n] = 0$

Covariance $K_W(k,j) = E[(U_0 + X_k)(U_0 + X_j)] = E[U_0^2] + E[X_k X_j] + 2E[U_0]E[X_k]$

$= 1 + \sigma^2 \delta_{k-j}$

$K_W(k) = 1 + \sigma^2 \delta_k$

PSD $S_W(f) = \delta(f) + \sigma^2$

Cross covariances

$K_{ZY} = E[Z_n Y_n] = E[X_n U_n (U_n + X_n)] = E[X_n U_n^2 + U_n X_n^2] = 0$

$K_{ZW} = E[Z_n W_n] = E[X_n U_n (U_0 + X_n)] = E[X_n U_n U_0 + X_n^2 U_0] = 0$

$K_{YW} = E[(U_n + X_n)(U_0 + X_m)] = E[U_n U_0 + \cancel{X_n U_0} + \cancel{U_n X_m} + X_n X_m]$

$= 1\delta_n + \sigma^2 \delta_{n-m}$

5.13) $S_X(f) = \frac{N_0}{2} \overset{\mathcal{F}}{\Rightarrow} R_X(t) = \frac{N_0}{2}\delta(t)$ To have a PSD, it must be weakly stationary.

$h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(f)$

$g(t) \overset{\mathcal{F}}{\longleftrightarrow} G(f)$

$$Y(t) = \int_0^\infty h(\tau)X(t-\tau)\,d\tau$$

$$V(t) = \int_0^\infty g(\tau)X(t-\tau)\,d\tau$$

a) $R_{Y,V}(t,s) = E(Y_t V_s)$

$$= E\left[\int_0^\infty h(\tau_t)X(t-\tau_t)d\tau_t \int_0^\infty g(\tau_s)X(s-\tau_s)\,d\tau_s\right]$$

$$= E\left[\int_0^\infty\int_0^\infty h(\tau_t)g(\tau_s)X(t-\tau_t)X(s-\tau_s)\,d\tau_t\,d\tau_s\right]$$

$$= \int_0^\infty\int_0^\infty h(\tau_t)g(\tau_s)E[X(t-\tau_t)X(s-\tau_s)]\,d\tau_t\,d\tau_s$$

$$= \int_0^\infty\int_0^\infty h(\tau_t)g(\tau_s)R_X(t-\tau_t, s-\tau_s)\,d\tau_t\,d\tau_s$$

$$= \int_0^\infty\int_0^\infty h(\tau_t)g(\tau_s)R_X(t-\tau_t-s+\tau_s)\,d\tau_t\,d\tau_s$$

$$= \frac{N_0}{2}\int_0^\infty\int_0^\infty h(\tau_t)g(\tau_s)\,\delta(t-\tau_t-s+\tau_s)\,d\tau_t\,d\tau_s$$

$$= \frac{N_0}{2}\int_0^\infty h(\tau_t)g(s-t+\tau_t)\,d\tau_t$$

$$= \frac{N_0}{2}\int_0^\infty h(\tau_t)g(\tau_t-(t-s))\,d\tau_t$$

If $E(Y_t V_s) = 0$, then Y_t and V_s are orthogonal, so uncorrelated, and because they are Gaussian, then independent. This integral is the convolution of h and time-reversed g, which results in a Fourier transform of $H(f)G(-f)$.

$$\int d\tau_t\, h(\tau_t)g(\tau_t-(t-s)) = \int H(f)G(-f)e^{j2\pi f(t-s)}df$$

This will be zero if $H(f)G(-f) = 0$ for all f.

You can see this is the case if you consider that if any spectral components are shared between $H(f)$ and $G(-f)$, then there will be information that is shared between $Y(t)$ and $V(t)$.

5.14 $\{X(t)\}$ and $\{Y(t)\}$ be zero-mean stationary Gaussian with $R(\tau)$ and $S(f)$.

$X(t)$ and $Y(t)$ independent

$E[X(t)Y(s)] = 0$, all t, s.

$\sigma^2 = R(0)$.

$$W(t) = X(t)\cos(2\pi f_0 t) + Y(t)\sin(2\pi f_0 t)$$

$$E[W(t)] = E[X(t)\cos(2\pi f_0 t) + Y(t)\sin(2\pi f_0 t)]$$

$$= E[X(t)]\cos(2\pi f_0 t) + E[Y(t)]\sin(2\pi f_0 t)$$

$$= 0$$

$$R_W(t,s) = E[W(t)W(s)]$$

$$= E\left[\left(X(t)\cos(2\pi f_0 t) + Y(t)\sin(2\pi f_0 t)\right)\left(X(s)\cos(2\pi f_0 s) + Y(s)\sin(2\pi f_0 s)\right)\right]$$

$$= E\left[(X(t)X(s))\right]\cos(2\pi f_0 t)\cos(2\pi f_0 s) + 0 + 0 + E[Y(t)Y(s)]\sin(2\pi f_0 t)\sin(2\pi f_0 s)$$

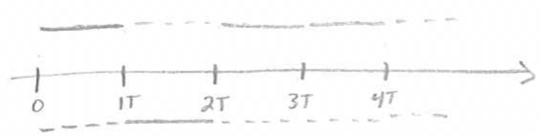
$$= R(t-s)\left[\cos(2\pi f_0 t)\cos(2\pi f_0 s) + \sin(2\pi f_0 t)\sin(2\pi f_0 s)\right]$$

$$= R(t-s)\left[\cos(2\pi f_0 (t-s))\right]$$

$\{W(t)\}$ is weakly stationary because the mean and correlation only depend on the time difference.

5.15) PAM Process

$X(t) = X_n ; \; t \in [(n-1)T, nT]$



Example waveform for PAM

$X(t) = \sum_k X_k \, p(t-kT)$

$Y(t) = X(t+Z)$ where Z is random variable, uniformly distributed on $[0, T]$

$E[Y(t)] = E[X(t+Z)] = 1(\frac{1}{2}) + (-1)(\frac{1}{2}) = 0$

$R_X(t,s) = E[Y(t)Y(s)] = E[X(t+Z)X(s+Z)]$

If $|t-s| > T$, then $E[X(t+Z)X(s+Z)] = \frac{1}{4}(1)(1) + \frac{1}{4}(1)(-1) + \frac{1}{4}(-1)(1) + \frac{1}{4}(-1)(-1)$
$= 0$

If $|t-s| \leq T$, we need to know the probability of overlap. When they overlap $X(t+Z)X(s+Z) = 1$. They are both either one or negative one.

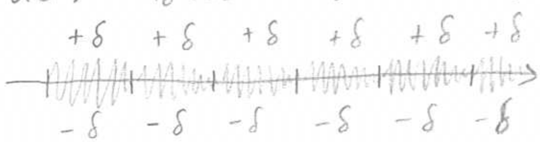
$Pr(\text{overlap}) = 1 - \frac{|t-s|}{T}$

$\therefore E[X(t+Z)X(s+Z)] = (1)(1 - \frac{|t-s|}{T}) = 1 - \frac{|t-s|}{T}$

$$R_X(t,s) = \begin{cases} 1 - \frac{|t-s|}{T}, & |t-s| \leq T \\ 0, & |t-s| > T \end{cases}$$

PSK Process

$U(t) = a_0 \cos(2\pi f_0 t + \delta X(t))$



The phase is shifted in each time block

$V(t) = U(t+\Theta)$ where Θ is a random variable uniformly distributed on $[0, \frac{1}{f_0}]$.

$E[V(t)] = E[U(t+\Theta)] = E[a_0 \cos(2\pi f_0(t+\Theta) + \delta X(t+\Theta))]$

$= a_0 E[\cos(2\pi f_0(t+\Theta))\cos(\delta X(t+\Theta)) - \sin(2\pi f_0(t+\Theta))\sin(\delta X(t+\Theta))]$

If $X(t+\Theta)$ stays the same throughout, then

$\int_0^{1/f_0} \cos(2\pi f_0(t+\Theta))\,d\Theta = 0$

If $\Phi(t+\theta)$ changes, then it is equally likely to go from $+1$ to -1 as it is to go from -1 to $+1$.

Thus,

$$a_0 E\Big[\cos(2\pi f_0(t+\theta))\overbrace{\cos(\delta\Phi(t+\theta))}^{\text{even function so stays constant}} - \sin(2\pi f_0(t+\theta))\sin(\delta\Phi(t+\theta))\Big]$$

$$= \frac{1}{2}\left[\int_0^a -\sin(2\pi f_0(t+\theta))\sin(\delta)\,d\theta + \int_a^{1/f_0} -\sin(2\pi f_0(t+\theta))\sin(-\delta)\,d\theta\right]$$

$$+ \frac{1}{2}\left[\int_0^a -\sin(2\pi f_0(t+\theta))\sin(-\delta)\,d\theta + \int_a^{1/f_0} -\sin(2\pi f_0(t+\theta))\sin(\delta)\,d\theta\right]$$

$$= 0$$

$\therefore E[V(t)] = 0.$

$$R_V(t,s) = E[V(t)V(s)] = E[U(t+\theta)U(s+\theta)]$$

$$= E\big[a_0\cos(2\pi f_0(t+\theta) + \delta\Phi(t+\theta))\,a_0\cos(2\pi f_0(s+\theta) + \delta\Phi(s+\theta))\big]$$

$$= \frac{a_0^2}{2}E\Big[\cos(2\pi f_0(t+\theta) - 2\pi f_0(s+\theta) + \delta\Phi(t+\theta) - \delta\Phi(s+\theta)) + \cos(2\pi f_0(t+\theta+s+\theta) + \delta\Phi(t+\theta) + \delta\Phi(s+\theta))\Big]$$

$$= \frac{a_0^2}{2}E\Big[\cos(2\pi f_0(t-s) + \delta[\Phi(t+\theta) - \Phi(s+\theta)]) + \cancel{\cos(2\pi f_0(t+s+2\theta) + \delta(\Phi(t+\theta) + \Phi(s+\theta)))}\Big] \rightarrow 0 \text{ for same reasons above}$$

$$= \frac{a_0^2}{2}E\Big[\cos(2\pi f_0(t-s) + \delta[\Phi(t+\theta) - \Phi(s+\theta)])\Big]$$

If $|t-s| > T$, then

$$= \frac{a_0^2}{2}\left[\frac{1}{4}\cos(2\pi f_0(t-s) + 2\delta) + \frac{1}{2}\cos(2\pi f_0(t-s)) + \frac{1}{4}\cos(2\pi f_0(t-s) - 2\delta)\right]$$

$$= \frac{a_0^2}{2}\left[\frac{1}{2}\cos(2\pi f_0(t-s))\cos(2\delta) + \frac{1}{2}\cos(2\pi f_0(t-s))\right]$$

$$= \frac{a_0^2}{4}\left[\cos(2\pi f_0(t-s))(1+\cos(2\delta))\right]$$

If $|t-s| \le T$, then

$$= \frac{a_0^2}{2}\left(1 - \frac{|t-s|}{T}\right)\cos(2\pi f_0(t-s)) + \frac{|t-s|}{T}\frac{a_0^2}{4}\left[\cos(2\pi f_0(t-s))(1+\cos(2\delta))\right]$$

This results in an autocorrelation function

$$R_V(t,s) = \begin{cases} \frac{a_0^2}{2}\left(1 - \frac{|t-s|}{T}\right)\cos(2\pi f_0(t-s)) + \frac{|t-s|}{T}\left(\frac{a_0^2}{4}\right)\cos(2\pi f_0(t-s))(1+\cos(2\delta)) & , |t-s| \le T \\ \frac{a_0^2}{4}\cos(2\pi f_0(t-s))(1+\cos(2\delta)) & , |t-s| > T \end{cases}$$

Notice that $R_X(t,s)$ and $R_V(t,s)$ are dependent only on the time difference, i.e. they are weakly stationary. Thus, we could calculate the PSD from the Fourier transform of the autocorrelation

5.29) $\{X_n\}, \{Z_n\}$ are zero-mean, mutually independent, iid, two-sided Gaussian random processes

$R_X(k) = \sigma_X^2 \delta_k$; $R_Z(k) = \sigma_Z^2 \delta_k$

$Y_n = Z_n + r Y_{n-1}$

$U_n = X_n + Z_n$

$W_n = U_n + r U_{n-1}$

$$K_U(n,k) = E[(U_n - \bar{U}_n)(U_k - \bar{U}_k)]$$
$$= E[(X_n + Z_n - \bar{X}_n - \bar{Z}_n)(X_k + Z_k - \bar{X}_k - \bar{Z}_k)]$$
$$= K_X(n,k) + K_Z(n,k) + E[(X_n - \bar{X}_n)(Z_k - \bar{Z}_k)] + E[(X_k - \bar{X}_k)(Z_n - \bar{Z}_n)]$$
$$= K_X(n,k) + K_Z(n,k) + \cancel{E[X_n - \bar{X}_n]}^{0} \cancel{E[Z_k - \bar{Z}_k]}^{0} + \cancel{E[X_k - \bar{X}_k]}^{0} \cancel{E[Z_n - \bar{Z}_n]}^{0}$$
$$= R_X(n,k) + R_Z(n,k)$$
$$= \sigma_X^2 \delta(n-k) + \sigma_Z^2 \delta(n-k) = \sigma_x^2 + \sigma_z^2 \delta(n-k)$$

$$S_U = \sigma_x^2 + \sigma_z^2$$

$$K_W(n,k) = E[(W_n - \bar{W}_n)(W_k - \bar{W}_k)]$$
$$= E[(U_n + r U_{n-1} - \bar{U}_n - r\bar{U}_{n-1})(U_k + r U_{k-1} - \bar{U}_k - r\bar{U}_{k-1})]$$
$$= E[(U_n - \bar{U}_n)(U_k - \bar{U}_k)] + r^2 E[(U_{n-1} - \bar{U}_{n-1})(U_{k-1} - \bar{U}_{k-1})]$$
$$+ E[(U_n - \bar{U}_n)(U_{k-1} - \bar{U}_{k-1})] + r^2 E[(U_{n-1} - \bar{U}_{n-1})(U_k - \bar{U}_k)]$$
$$= K_U(n,k) + r^2 K_U(n-1,k-1) + K_U(n,k-1) + r^2 K_U(n-1,k)$$
$$= (\sigma_X^2 + \sigma_Z^2)(\delta(n-k) + r^2 \delta(n-k) + \delta(n-k+1) + r^2 \delta(n-1-k))$$
$$= (\delta_x^2 + \sigma_z^2)(\delta(n-k) + r^2 \delta(n-k) + \delta(n-k+1) + r^2 \delta(n-k-1))$$

$$K_W(k) = (\delta_x^2 + \delta_z^2) = (\delta(k) + r^2 \delta(k) + \delta(k+1) + r^2 \delta(k-1))$$

$$S_W(f) = (\sigma_x^2 + \delta_z^2)(1 + r^2 + e^{-j2\pi f(-1)} + r^2 e^{-j2\pi f(+1)})$$

$$E[(X_n - W_n)^2] = E[(X_n - U_n - rU_{n-1})^2]$$

$$= E[(X_n - (X_n - Z_n) - r(X_{n-1} - Z_{n-1}))^2]$$

$$= E[(Z_n - rX_{n-1} + rZ_{n-1})^2]$$

$$= E[Z_n^2 - \cancel{rZ_nX_{n-1}} + \cancel{rZ_nZ_{n-1}} - \cancel{rZ_nX_{n-1}} + r^2X_{n-1}^2 - \cancel{r^2X_{n-1}Z_{n-1}} + \cancel{rZ_nZ_{n-1}}$$
$$- \cancel{r^2X_{n-1}Z_{n-1}} + r^2Z_{n-1}^2]$$

$$= E[Z_n^2 + r^2X_{n-1}^2 + r^2Z_{n-1}^2]$$

$$= \sigma_Z^2 + r^2\sigma_X^2 + r^2\sigma_Z^2$$

$$= \underline{r^2\sigma_X^2 + (1+r^2)\sigma_Z^2}$$

5.30) $\{Z_n\}$ and $\{W_n\}$ are two mutually independent two-sided zero-mean iid Gaussian processes with variances σ_Z^2 and σ_W^2.

$$X_n = Z_n - r Z_{n-1} \quad \text{with } 0 < r < 1.$$

$$Y_n = X_n + W_n$$

$$U_n = r U_{n-1} + Y_n$$

a) Z_n is iid. $R_Z(k) = \sigma_Z^2 \delta_k \therefore S_Z(f) = \sigma_Z^2$ all f

b) $$R_X(k) = E[(Z_n - rZ_{n-1})(Z_{n+k} - rZ_{n+k-1})]$$
$$= R_Z(k) - rR_Z(k-1) - rR_Z(k+1) + r^2 R_Z(k)$$
$$= \sigma_Z^2 \left[(1+r^2)\delta_k - r(\delta_{k-1} + \delta_{k+1})\right]$$

$$S_X(f) = \sigma_Z^2 \left((1+r^2) - r\left(e^{-j2\pi f(+1)} + e^{-j2\pi f(-1)}\right)\right)$$
$$= \sigma_Z^2 \left(1 + r^2 - 2r\cos(2\pi f)\right)$$

c) $$R_Y(k) = E[Y_n Y_{n+k}]$$
$$= E[(X_n + W_n)(X_{n+k} + W_{n+k})]$$
$$= R_X(k) + \cancel{E[W_n X_{n+k}]}^{\to 0} + \cancel{E[X_n W_{n+k}]}^{\to 0} + R_W(k)$$
$$= \sigma_Z^2 \left[(1+r^2)\delta_k - r(\delta_{k-1} + \delta_{k+1})\right] + \sigma_W^2 \delta_k$$

$$S_Y(f) = \sigma_Z^2 \left(1 + r^2 - 2r\cos(2\pi f)\right) + \sigma_W^2$$

d) $E[(U_n - Z_n)^2]$

$$U_n = rU_{n-1} + X_n + W_n$$

$$U_n = rU_{n-1} + Z_n - rZ_{n-1} + W_n$$

$$U_n - Z_n = rU_{n-1} + Z_n - rZ_{n-1} + W_n - Z_n$$

$$U_n - Z_n = rU_{n-1} - rZ_{n-1} + W_n$$

from above

$$U_n = r U_{n-1} + X_n + W_n$$

$$= r(r U_{n-2} + X_{n-1} + W_{n-1}) + X_n + W_n$$

$$= r(r(rU_{n-3} + X_{n-2} + W_{n-2}) + X_{n-1} + W_{n-1}) + X_n + W_n$$

$$= r(r(r(rU_{n-4} + X_{n-3} + W_{n-3}) + X_{n-2} + W_{n-2}) + X_{n-1} + W_{n-1}) + X_n + W_n$$

$$= r^4 U_{n-4} + r^3 X_{n-3} + r^3 W_{n-3} + r^2 X_{n-2} + r^2 W_{n-2} + r X_{n-1} + r W_{n-1} + X_n + W_n$$

$$= r^k U_{n-k} + \sum_{i=0}^{k} r^i X_{n-i} + \sum_{i=0}^{k} r^i W_{n-i}$$

$$= r^k U_{n-k} + \sum_{i=0}^{k} r^i (Z_{n-i} - r Z_{n-i-1}) + \sum_{i=0}^{k} r^i W_{n-i}$$

This is a two-sided process so $k \to \infty$

$$U_n = \sum_{i=0}^{\infty} r^i (Z_{n-1} - r Z_{n-i-1}) + \sum_{i=0}^{\infty} r^i W_{n-i}$$

$$= \sum_{i=0}^{\infty} r^i Z_{n-1} - r \sum_{i=0}^{\infty} r^i Z_{n-i-1} + \sum_{i=0}^{\infty} r^i W_{n-i}$$

$$U_n - Z_n = \sum_{i=0}^{\infty} r^i Z_{n-i} - r \sum_{i=0}^{\infty} r^i Z_{n-i-1} + \sum_{i=0}^{\infty} r^i W_{n-i} - Z_n$$

$$E[(U_n - Z_n)^2] = E\left[\left(\sum_{i=0}^{\infty} r^i Z_{n-i} - r \sum_{i=0}^{\infty} r^i Z_{n-i-1} + \sum_{i=0}^{\infty} r^i W_{n-i} - Z_n\right)^2\right]$$

$$= E\left[\left(\sum_{i=0}^{\infty} r^i Z_{n-i} - \sum_{i=0}^{\infty} r^i Z_{n-i}\right) + \sum_{i=0}^{\infty} r^i W_{n-i}\right)^2\Bigg]$$

Because W are mutually independent

$$= E\left[\left(\sum_{i=0}^{\infty} r^i W_{n-i}\right)^2\right] = \sum_{i=0}^{\infty} r^{2i} E[W_n^2] = \boxed{\frac{\sigma_w^2}{1 - r^2}}$$