ECEn 670

Homework Problem Set 7

Due at beginning of class, Thursday, December 10, 2009

Problems are from *An Introduction to Statistical Signal Processing* by Gray and Davisson unless otherwise specified.

1. 6.1
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6.1) $Y_n = \begin{cases} 0 & n = 0 \\ \sum_{i=1}^{n} X_i, & n = 1, 2, \ldots \end{cases}$

$\{X_n\}$ is iid with Poisson marginal pmf with parameter λ.

Find: pmf for Y_n

Find $\sigma^2_{Y_n}$, EY_n, $K_Y(t,s)$

$$M_{Y_n}(ju) = E\left[e^{ju\sum_i X_i}\right] = \prod_i E\left[e^{juX_i}\right]$$

$$= M_X(ju)^n = e^{\lambda n(e^{ju}-1)}$$

This is Poisson with parameter $n\lambda$

$$P_{Y_n}(y) = \frac{(n\lambda)^k}{k!} e^{-n\lambda}; \; k = 0, 1, 2 \ldots$$

$EY_n = n\,EX = n\lambda$

$K_Y(t,s) = \sigma^2_X \min(t,s); \; t,s = 1, 2, \ldots$ (see page 371)

$= \lambda \min(t,s); \; t,s = 1, 2, \ldots$

$\sigma^2_{Y_n} = K_Y(n,n) = n\,\sigma^2_X = n\lambda$

6.4) $\{X_n\}$ with

$$P_{X_n}(+1) = P_{X_n}(-1) = \epsilon/2$$

$$P_{X_n}(0) = 1-\epsilon$$

$$Y_n = \frac{1}{N}\sum_{i=0}^{N-1} X_{n-i}$$

a) $$EX_n = (1)(\tfrac{\epsilon}{2}) + (-1)(\tfrac{\epsilon}{2}) + (0)(1-\epsilon) = 0$$

$$\sigma^2_{X_n} = E[(X_n - E[X_n])^2] = (1)^2(\tfrac{\epsilon}{2}) + (-1)^2(\tfrac{\epsilon}{2}) + (0)(1-\epsilon) = \epsilon$$

$$M_{X_n}(ju) = E[e^{juX_n}] = e^{ju(1)}(\tfrac{\epsilon}{2}) + e^{ju(-1)}(\tfrac{\epsilon}{2}) + e^{ju(0)}(1-\epsilon)$$

$$= (\tfrac{\epsilon}{2})(e^{ju}+e^{-ju}) + (1-\epsilon)$$

$$= \epsilon\cos(u) + (1-\epsilon)$$

$$K_{X_n}(t,s) = \begin{cases} \sigma_X^2 = \epsilon, & t = s \\ 0, & t \neq s \end{cases}$$ Because $\{X_n\}$ is iid.

b) $$EY_n = E\left[\frac{1}{N}\sum_{i=0}^{N-1} X_{n-i}\right] = \frac{1}{N}E\left[\sum_{i=0} X_{n-i}\right] = \frac{1}{N}(N)EX_i = \underline{0}$$

$$\sigma^2_{Y_n} = EY_n^2 = \frac{1}{N^2}E\left(\sum_{i=0}^{N-1} X_{n-i}\right)^2 = \frac{1}{N^2}\sum_{i=0}^{N-1} E(X_{n-i}^2) = \frac{1}{N^2}N(\epsilon) = \underline{\epsilon/N}$$

Because $E(X_{n-i}X_{n-j}) = 0$ except when $i=j$

$$M_{Y_n}(ju) = E\left[e^{ju\frac{1}{N}\sum_{i=0}^{N-1} X_{n-i}}\right] = \prod_i E\left[e^{ju\frac{1}{N}X_i}\right] = \left(M_X(ju\tfrac{1}{N})\right)^N$$

$$= \left(\epsilon\cos(\tfrac{u}{N}) + (1-\epsilon)\right)^N$$

c) $$R_{X,Y}(t,s) \triangleq E[X_t Y_s] = E\left[X_t \frac{1}{N}\sum_{i=0}^{N-1} X_{s-i}\right] = \frac{1}{N}\sum_{i=0}^{N-1} E[X_t X_{s-i}] = \frac{\epsilon}{N}\sum_{i=0}^{N-1}\delta_{t-s+i}$$

$$= \begin{cases} \frac{\epsilon}{N}, & \text{if } s-t \in \{0,\dots,N-1\} \\ 0, & \text{otherwise} \end{cases}$$

d) $$P_r(|Y_n| > \delta) \leq \frac{\sigma^2_{Y_n}}{\delta^2} = \frac{\epsilon}{N\delta^2}$$ By Chebyshev for $\delta > 0$.

6.7) $Pr(X(t)=1) = Pr(X(t)=0) = \frac{1}{2}$ for stationary CT process $\{X(t)\}$

In $(0,t]$, then

$$P_{N_t}(k) = \frac{1}{1+\alpha t}\left(\frac{\alpha t}{1+\alpha t}\right)^k \; ; \; k = 0, 1, 2, \ldots,$$

where $\alpha > 0$

a) $$M_{N_t}(ju) = E[e^{juN_t}]$$
$$= \sum_{k=0}^{\infty} \frac{1}{1+\alpha t}\left(\frac{\alpha t}{1+\alpha t}\right)^k e^{juk}$$
$$= \frac{1}{1+\alpha t}\sum_{k=0}^{\infty}\left(\frac{\alpha t}{1+\alpha t}e^{ju}\right)^k$$
$$= \frac{1}{1+\alpha t}\;\frac{1}{1-\frac{\alpha t}{(1+\alpha t)}e^{ju}} = \frac{1}{1+\alpha t-\alpha t(e^{ju})} = \underline{(1+\alpha t-\alpha t e^{ju})^{-1}}$$

$$\frac{dM_{N_t}(ju)}{du} = (-1)(1+\alpha t-\alpha t e^{ju})^{-2}(-\alpha t e^{ju} j) =$$
$$= \underline{j\alpha t e^{ju}(1+\alpha t-\alpha t e^{ju})^{-2}}$$

Setting $u=0$, $\frac{dM_{N_t}(ju)}{du} = j\alpha t(1+\alpha t-\alpha t)^{-2} = j\alpha t$

$$E(N_t) = \frac{j\alpha t}{j} = \underline{\alpha t}$$

$$\frac{d^2 M_{N_t}(ju)}{du} = -\alpha t e^{ju}(1+\alpha t-\alpha t e^{ju})^{-2} + j\alpha t e^{ju}(-2)(1+\alpha t-\alpha t e^{ju})^{-3}(-\alpha t e^{ju} j)$$

with $u=0$
$$\frac{d^2 M_{N_t}(j0)}{du} = -\alpha t + j\alpha t(-2)(-j\alpha t) = -\alpha t - 2(\alpha t)^2$$

$$E[N_t^2] = \frac{-\alpha t - 2(\alpha t)^2}{-1} = \alpha t + 2(\alpha t)^2$$

$$\sigma_{N_t}^2 = E[N_t^2] - (E[N_t])^2 = \alpha t + 2(\alpha t)^2 - (\alpha t)^2 = \alpha t + (\alpha t)^2$$

b) $E X_t = 1(1/2) + 0(1/2) = 1/2$

$R_X(t,s) = E[X_t X_s] = 1 \times Pr(X_t = X_s = 1)$

Suppose $t > s$.

$$R_X(t,s) = Pr(X_s = 1) \times Pr(X_t = 1 \mid X_s = 1)$$

$$= Pr(N_{t-s} \text{ even})$$

$$Pr(N_t \text{ even}) = \sum_{k \text{ even}} \frac{1}{1+\alpha t} \left(\frac{\alpha t}{1+\alpha t}\right)^k$$

$$= \sum_{k=0}^{\infty} \frac{1}{1+\alpha t} \left(\frac{\alpha t}{1+\alpha t}\right)^{2k} = \frac{1}{1+\alpha t} \frac{1}{1-\left(\frac{\alpha t}{1+\alpha t}\right)^2}$$

$$= \frac{1+\alpha t}{(1+\alpha t)^2 - (\alpha t)^2} = \frac{1+\alpha t}{1+2\alpha t + (\alpha t)^2 - (\alpha t)^2} = \frac{1+\alpha t}{1+2\alpha t}$$

For $t > s$,

$$R_X(t,s) = \frac{1}{2} \frac{1+\alpha(t-s)}{1+2\alpha(t-s)}$$

By symmetry

$$R_X(\tau) = \frac{1}{2} \frac{1+\alpha|\tau|}{1+2\alpha|\tau|}$$

6.11) $\{X(t)\}$ is CT zero-mean, stationary Gaussian with $R(\tau)$, $S(f)$

$\{Y(t)\}$ is CT zero-mean, stationary Gaussian $R(\tau)$, $S(f)$

$X(t), Y(t)$ independent

$E[X(t)Y(s)] = 0$, all t, s,

$\sigma^2 = R(0)$

$W(t) = X(t)\cos(2\pi f_0 t) + Y(t)\sin(2\pi f_0 t)$

$W(t)$ is Gaussian since it is formed from simple linear combinations of Gaussian random variables $X(t)$ and $Y(t)$.

Gaussian processes are completely characterized by means and covariances.

$\rightarrow f_{W(t)}$ is Gaussian with mean 0 and variance $R_W(t,t) = R(0) = \sigma^2$ (because zero mean)

$\rightarrow f_{W(t),W(s)}(u,v)$ is Gaussian with mean vector of $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and

covariance $K_W(u,v) = R_W(u,v) = R(u-v)\left[\cos(2\pi f_0(t-s))\right]$

$\rightarrow \{W(t)\}$ is a Gaussian process

$\rightarrow$ It is strictly stationary because it is weakly stationary and Gaussian

6.12) $Y_n = (-1)^{N_n}$

$Y_n = e^{j\pi N_n}$

$\rightarrow E[Y_n] = E[e^{j\pi N_n}] = M_{N_n}(j\pi) =$

$= ((1-p) + pe^{j\pi})^n = (1-2p)^n$

$R_Y(n,k) = E[Y_n Y_k] = E[e^{j\pi N_n} e^{j\pi N_k}] = E[e^{j\pi(N_n+N_k)}]$

Suppose $n > k$

$$N_n = \sum_{i=1}^{n} X_i$$

$$N_n + N_k = \sum_{i=1}^{n} X_i + \sum_{i=1}^{k} X_i = 2\sum_{i=1}^{k} X_i + \sum_{i=k+1}^{n} X_i$$

$$R_Y(n,k) = E\left[e^{ju\,2\sum_{i=1}^{k} X_i} e^{ju\sum_{i=k+1}^{n} X_i}\right]$$

$$= M_X(j2\pi)^k \, M_X(j\pi)^{n-k}$$

$$= \underbrace{((1-p) + pe^{j2\pi})}_{1}((1-p) + pe^{j\pi})^{n-k}$$

$$= (1-2p)^{n-k}$$

By symmetry

$\rightarrow R_Y(k) = (1-2p)^{|k|}$

This is technically not weakly stationary because of $E[Y_n] = (1-2p)^n$

In the limit,

$$S(f) = \sum_k (1-2p)^{|k|} e^{-j2\pi f k}$$

$$Pr(Y_n = y \mid Y_{n-1} = y_{n-1}, Y_{n-2} = y_{n-2}, \ldots, Y_0 = y_0) = \begin{cases} Pr(X_n = 1): & y_n \neq y_{n-1} \\ Pr(X_n = 0): & y_n = y_{n-1} \end{cases} = Pr(Y_n = y \mid Y_{n-1} = y_{n-1})$$

so this is Markov

6.14) (U,W) is Gaussian with $EU = EW = 0$

$E(U^2) = E(W^2) = \sigma^2$

$E(UW) = \rho\sigma^2$

$S = U + W$

$D = U - W$

a) S and D will be Gaussian

$ES = E(U+W) = 0$

$ED = E(U-W) = 0$

$$\sigma_S^2 = E[(U+W)^2] = E(U^2) + E(W^2) + 2E[UW] = \sigma^2 + \sigma^2 + 2\rho\sigma^2$$
$$= 2\sigma^2(1+\rho)$$

$$\sigma_D^2 = E[(U-W)^2] = E(U^2) + E(W^2) - 2E(UW) = \sigma^2 + \sigma^2 - 2\rho\sigma^2$$
$$= 2\sigma^2(1-\rho)$$

$$f_S(s) = \frac{e^{-s^2/2\sigma_S^2}}{\sqrt{2\pi\sigma_S^2}} = \frac{e^{-s^2/4\sigma^2(1+\rho)}}{\sqrt{4\pi\sigma^2(1+\rho)}}$$

$$f_D(d) = \frac{e^{-s^2/2\sigma_D^2}}{\sqrt{2\pi\sigma_D^2}} = \frac{e^{-s^2/4\sigma^2(1-\rho)}}{\sqrt{4\pi\sigma^2(1-\rho)}}$$

b) Mean vector $m = \begin{bmatrix} ES \\ ED \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Lambda = \begin{bmatrix} E(S^2) & E(SD) \\ E(SD) & E(D^2) \end{bmatrix} = \begin{bmatrix} 2\sigma^2(1+\rho) & 0 \\ 0 & 2\sigma^2(1-\rho) \end{bmatrix}$$

$$E[SD] = E[(U+W)(U-W)] = E[U^2 - W^2] = E[U^2] - E[W^2] = 0$$

Since they are uncorrelated, they are independent.

$$f_{S,D}(s,d) = f_S(d) f_D(d)$$

6.15) a) $Pr(N=n) = Pr(K=n-1) = \dfrac{\lambda^{n-1} e^{-\lambda}}{(n-1)!}$, $n = 1, 2, \ldots$

$$E[N] = E[K+1] = E[K] + 1 = \lambda + 1$$

$$M_N(ju) = E[e^{ju(K+1)}] = e^{ju} M_K(ju)$$
$$= e^{ju} e^{\lambda(e^{ju}-1)}$$

Notice that if we had

$N = K + b$ we would end up with

$$M_N(ju) = e^{jub} e^{\lambda(e^{ju}-1)}$$

T_i have characteristic function M_N

$$S_k = \sum_{i=1}^{k} T_i$$

$$M_{S_k}(ju) = M_N(ju)^k = e^{juk} e^{\lambda k (e^{ju}-1)}$$

This is just a shift of k above of a Poisson process with parameter λk

S_K is the sum of iid random variables and thus has independent increments

6.22) $\{N_t ; t \geq 0\}$ is isi process

$$P_{N_t}(k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \; ; k = 0, 1, 2, \ldots$$

$\{L_t , t \geq 0\}$ is isi process

$$P_{L_t}(k) = \frac{(\nu t)^k e^{-\nu t}}{k!} \; ; k = 0, 1, 2, \ldots$$

N_t and L_t mutually independent

$$I_t = N_t + L_t$$

a) $M_{I_t}(ju) = M_{N_t}(ju) \, M_{L_t}(ju)$

$$= e^{\lambda t (e^{ju} - 1)} \, e^{\nu t (e^{ju} - 1)}$$

$$= e^{(\lambda + \nu) t (e^{ju} - 1)}$$

$$P_{I_t}(k) = \frac{[(\lambda + \nu) t]^k e^{-(\lambda + \nu) t}}{k!} \; ; k = 0, 1, 2, \ldots$$

Merging of two Poisson processes

b) $E[I_t] = \lambda + \nu$

$K_{I_t}(t, s) = (\lambda + \nu) \min(t, s) \quad ; t, s \geq 0$

c) $\{I_t ; t \geq 0\}$ is an isi process because it is made up of two isi processes

d) $P_r(Z = N_t) = \sum_{k=j} P_Z(k) \, P_{N_t}(j)$

$$= \sum_{k=j} \left(\frac{a^k}{(1+a)^{k+1}} \right) \frac{(\lambda t)^j e^{-\lambda t}}{j!} \qquad ; k, j = 0, 1, 2, \ldots$$

$$= \frac{e^{-\lambda t}}{1+a} \sum_{k=j} \left(\frac{a}{1+a} \right)^k \frac{(\lambda t)^k}{k!} = \frac{e^{-\lambda t}}{1+a} \sum_{k=j} \frac{\left(\frac{a \lambda t}{1+a} \right)^k}{k!} \quad ; k = 0, 1, 2, \ldots$$

$$= \frac{e^{-\lambda t}}{1+a} e^{\left(\frac{\lambda t a}{1+a} \right)} = \frac{e^{\left(\frac{-\lambda t - \lambda t a + \lambda t a}{1+a} \right)}}{1+a} = \frac{e^{\frac{-\lambda t}{1+a}}}{1+a}$$

e) $\quad Y_t = \sum_{k=0}^{N_t} Z_k$

$$P_Z(k) = \frac{a^k}{(1+a)^{k+1}} = \left(\frac{a}{1+a}\right)^k \frac{1}{1+a}$$

$$E[Z] = \sum_k \left(\frac{a}{1+a}\right)^k \frac{1}{1+a} k = \frac{1}{1+a} \sum_k k\left(\frac{a}{1+a}\right)^k$$

$$= \frac{1}{(1+a)} \frac{a}{(1+a)} \sum_k k\left(\frac{a}{1+a}\right)^{k-1}$$

$$= \frac{a}{(1+a)^2} \cdot \frac{1}{\left(1-\frac{a}{1+a}\right)^2}$$

$$= \frac{a}{(1+a-a)^2} = a$$

$$E[Y_t] = E\Big(E[Y_t \mid N_t]\Big) = E\big[(N_t+1)\, E[Z]\big]$$

$$= (\lambda t + 1)\, a$$

$$E[Z^2] = \frac{1}{1+a} \sum_k k^2 \left(\frac{a}{1+a}\right)^k$$

$$= \frac{1}{1+a} \frac{a}{1+a} \sum_k k^2 \left(\frac{a}{1+a}\right)^{k-1}$$

$$= \frac{a}{(1+a)^2}\left(\frac{2\left(\frac{a}{1+a}\right)}{\left(1-\left(\frac{a}{1+a}\right)\right)^3} + \frac{1}{\left(1-\left(\frac{a}{1+a}\right)\right)^2}\right)$$

$$= \frac{a}{1}\left(\frac{2a}{(1+a-a)} + \frac{1}{(1+a-a)^2}\right)$$

$$= 2a^2 + a$$

$$\sigma_Z^2 = 2a^2 + a - a^2$$

$$= a^2 + a = \underline{a(a+1)}$$

$$\sigma_{Y_t}^2 = E\left[(Y_t - EY_t)^2\right]$$

$$= E\left[\left(\sum_{k=0}^{N_t} Z_k - \sum_{k=0}^{N_t} EZ\right)^2\right] = E\left[\left(\sum_{k=0}^{N_t} Z_k - EZ\right)^2\right]$$

$$= E\left[E\left(\left[\sum_{k=0}^{N_t} Z_k - EZ\right]^2 \,\Big|\, N_t\right)\right]$$

$$= E\left[E\left(\left(\sum_{k=0}^{N_t} Z_k - EZ\right)\left(\sum_{j=0}^{N_t} Z_j - EZ\right) \,\Big|\, N_t\right)\right]$$

$$E\big[(Z_1 - EZ)(Z_2 - EZ)\big] = E\big(Z_1 Z_2 - (EZ)Z_2 - Z_1(EZ) + (EZ)^2\big) = 0$$

so only matters when $k=j$

$\therefore$

$$= E\left[(N_t + 1)\, \sigma_Z^2\right]$$

$$= \underline{(\lambda t + 1)\, a(a+1)}$$

6.26) N sensors

$$W_i = X + Y_i \quad , \; i = 0, 1, 2, \ldots, N-1$$

with $X, Y_0, \ldots, Y_{N-1}$ all independent Gaussian with zero mean

$$\sigma_X^2 = 1 \quad , \quad \sigma_{Y_i}^2 = r^i \text{ for } |r| < 1$$

$$\hat{X}_N = \frac{1}{N} \sum_{i=0}^{N-1} W_i$$

(a) $$\hat{X}_N = \frac{1}{N} \sum_{i=0}^{N-1} W_i = \frac{1}{N} \sum_{i=0}^{N-1} X + Y_i = X + \frac{1}{N} \sum_{i=0}^{N-1} Y_i$$

$$E[\hat{X}_N] = EX + \frac{1}{N} \sum_{i=0}^{N-1} EY_i = 0$$

$$\sigma_{\hat{X}}^2 = \sigma_X^2 + \frac{1}{N^2} \sum_{i=0}^{N-1} \sigma_{Y_i}^2 = 1 + \frac{1}{N^2} \sum_{i=0}^{N-1} r^i$$

$$= 1 + \frac{1}{N^2} \frac{1 - r^N}{1 - r}$$

$\hat{X}_N$ is Gaussian because it is formed by a linear combination of Gaussians. The pdf is thus defined by the mean, $E[\hat{X}_N] = 0$ and the variance $\sigma_{\hat{X}}^2 = 1 + \frac{1}{N^2} \frac{1 - r^N}{1 - r}$

b) $$\epsilon_N = X - \hat{X}_N = X - \left(X + \frac{1}{N} \sum_{i=0}^{N-1} Y_i\right) = -\frac{1}{N} \sum_{i=0}^{N-1} Y_i$$

$$E[\epsilon_N] = E[X] - E[\hat{X}_N] = 0$$

$$\sigma_{\epsilon_N}^2 = E\left[(X - \hat{X}_N)^2\right] = E\left[\left(X - X - \frac{1}{N} \sum_{i=0}^{N-1} Y_i\right)^2\right] = \frac{1}{N^2} E\left[\left(\sum_{i=0}^{N-1} Y_i\right)^2\right]$$

$$= \frac{1}{N^2} \sum_{i=0}^{N-1} r^i = \boxed{\frac{1}{N^2} \frac{1 - r^N}{1 - r}}$$

c) $$\sigma_{\epsilon_N}^2 \to 0 \text{ as } N \to \infty$$

$\therefore \hat{X}_N \to X$ by Chebyshev Inequality

$$\lim_{N \to \infty} P(|X - \hat{X}_N| \geq \alpha) \leq \frac{\sigma_{\epsilon_N}^2}{\alpha^2}$$

$$\therefore P(|X - \hat{X}_N| \geq \alpha) \leq 0$$

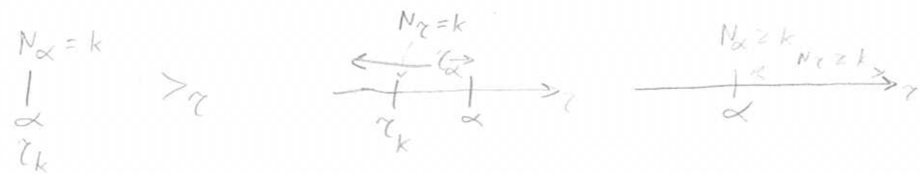
$\hat{X}_N$ converges in probability to the true value X.

6.29)

a) If $\tau_k = \alpha$, then $N_\alpha = k$ from definition of τ_k.

If $\tau_k < \alpha$, then suppose $\tau_\alpha = \tau < \alpha$, which means that $N_\tau = k$ for some $\tau < \alpha$, yielding $N_\alpha \geq k$.

If $N_\alpha \geq k$, then also $N_\tau \geq k$ for all $\tau \geq \alpha$ and hence $\tau_\alpha \leq \tau$ for all $\tau > \alpha$ and hence $\tau_k \leq \alpha$.



$$F_{\tau_k}(\alpha) = P_r(\tau_k \leq \alpha) = P_r(N_\alpha \geq k)$$
$$= \sum_{i=k}^{\infty} \frac{(\lambda\alpha)^i e^{-\lambda\alpha}}{i!}$$

Term by term

$$f_{\tau_k} = \frac{dF_{\tau_k}(\alpha)}{d\alpha}$$
$$= \sum_{i=k}^{\infty} \left(\frac{\lambda i (\lambda\alpha)^{i-1} e^{-\lambda\alpha}}{i!} - \frac{\lambda(\lambda\alpha)^i e^{-\lambda\alpha}}{i!} \right)$$
$$= \lambda e^{-\lambda\alpha} \sum_{i=k}^{\infty} \left(\frac{(\lambda\alpha)^{i-1}}{(i-1)!} - \frac{(\lambda\alpha)^i}{i!} \right)$$ only the smallest k term will remain

$$= \lambda e^{-\lambda\alpha} \frac{(\lambda\alpha)^{k-1}}{(k-1)!}, \quad k = 1, 2, \ldots$$

b) $\tau_n \leq \alpha$ iff $\tau_n + \tau_{n-1} \leq \tau_{n-1} + \alpha$ iff $N_{\tau_{n-1}+\alpha} \geq n$

iff $N_{\tau_{n-1}+\alpha} > n-1 = N_{\tau_{n-1}}$.

If $\tau_i = \beta_i$, $i = 1, 2, \ldots, n-1$, then

$$\tau_{n-1} = \sum_{i=1}^{n-1} \tau_i = \sum_{i=1}^{n-1} \beta_i = \beta_{n-1}$$

$$F_{\tau_n | \tau_{n-1}, \ldots, \tau_1}(\alpha | \beta_{n-1}, \ldots, \beta_1) = P_r(\tau_n \leq \alpha | \tau_{n-1} = \beta_{n-1}, \ldots, \tau_1 = \beta_1)$$
$$= P_r(N_{\beta_{n-1}+\alpha} > N_{\beta_{n-1}} | \tau_{n-1} = \beta_{n-1}, \ldots, \tau_1 = \beta_1)$$
$$= P_r(N_\alpha > N_0)$$ since stationary increments
$$= P_r(N_\alpha \geq 1) = 1 - e^{-\lambda\alpha}$$
$$= F_{\tau_n}(\alpha); \quad n = 1, 2, \ldots; \alpha \geq 0$$ They are iid.