ECEn 670

Quiz 4

Thursday, October 22, 2009

1. You are given two random variables $X$ and $Y$ with a joint pmf described by

$$p_{X,Y}(x,y) = \begin{cases} 
\frac{1}{2}, & \text{if } (x,y) = (0,0) \\
\frac{1}{6}, & \text{if } (x,y) = (1,1), (1,2), (2,2) \\
0, & \text{otherwise}
\end{cases}$$

$$E[X] = \frac{1}{2}(0) + \frac{1}{6}(1 + 1 + 2) = \frac{4}{6} = \frac{2}{3}$$

$$E[Y] = \frac{1}{2}(0) + \frac{1}{6}(1 + 2 + 2) = \frac{5}{6}$$

What is the correlation of $X$ and $Y$? 

$$E[XY] = \frac{1}{2}(0 \cdot 0) + \frac{1}{6}(1 \cdot 1 + 1 \cdot 2 + 2 \cdot 2) = 0 + \frac{1}{6}(1 + 2 + 4) = \frac{2}{3}$$

$$\text{var}(X) = E[X^2] - (E[X])^2 = \frac{1}{6}(1^2 + 1^2 + 2^2) - \left(\frac{2}{3}\right)^2 = \frac{5}{18} - \frac{4}{9} = \frac{1}{18}$$

$$\text{var}(Y) = E[Y^2] - (E[Y])^2 = \frac{5}{18} - \frac{25}{36} = \frac{5}{36}$$

$$\text{cov}(X,Y) = E[XY] - E[X]E[Y] = \frac{2}{3} - \frac{4}{9} = \frac{2}{9}$$

$$\rho_{XY} = \frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)\text{var}(Y)}} = \frac{\frac{2}{9}}{\sqrt{\frac{1}{18}\frac{5}{36}}} = \frac{\frac{2}{9}}{\frac{1}{18\sqrt{5}}} = \frac{4}{\sqrt{5}}$$


What is the correlation coefficient between $X$ and $Y$?

$$E[Y|X = 1] = \frac{\frac{1}{2}(1) + \frac{1}{6}(2)}{\frac{1}{2} + \frac{1}{6}} = \frac{3}{4}$$

$$P(Y|X = 1) = \frac{3}{4}$$

Given that for two Gaussian random variables $U$ and $V$, $m_{UV} = m_U + \rho_{UV} \sigma_U (u - m_U)$, what is the best affine estimator, $\hat{Y}[X]$, for $Y$ in the minimum mean squared error (MMSE) sense?

$$\begin{align*}
\hat{Y}[X] &= E[Y|X] \\
&= \frac{E[X]}{E[Y]} + \rho_{XY} \frac{E[Y]}{E[X]} (X - E[X]) \\
&= \frac{\frac{2}{3}}{\frac{5}{6}} + \frac{\frac{2}{3}}{\frac{5}{6}} \left( x - \frac{2}{3} \right)
\end{align*}$$

Without restriction, what is the best estimator, $\hat{Y}[X]$, for $Y$ in the minimum mean squared error (MMSE) sense?

$$\hat{Y}[X] = E[Y|X] = \begin{cases} 
0, & i f \ x = 0 \\
\frac{1}{2}, & i f \ x = 1 \\
2, & i f \ x = 2
\end{cases}$$