**Homework 1 Solutions**
ECEn 670, Fall 2013

***A.1. Use the first seven relations to prove relations (A.10), (A.13), and (A.16).***
Prove $(F \cup G)^c = F^c \cap G^c$ (A.10).
$(F \cup G)^c = ((F^c \cap G^c)^c)^c$ by A.6.
$(F \cup G)^c = F^c \cap G^c$ by A.4

Prove $F \cup (F \cap G) = F = F \cap (F \cup G)$ (A.13).
$F \cap (F \cup G) = (F \cap F) \cup (F \cap G)$ by A.3.
$F \cap (F \cup G) = F \cup (F \cap G)$ by A.20 (proved in book)
Now let's look at:
$F \subset F \cup X \therefore F \subset F \cup (F \cap G)$
$F \cap X \subset F \therefore F \cap (F \cup G) \subset F$
Because$F \cap (F \cup G) = F \cup (F \cap G)$,
$F \subset F \cap (F \cup G)$ and $F \cap (F \cup G) \subset F$
$\therefore F = F \cap (F \cup G)$
$F \cup (F \cap G) = F = F \cap (F \cup G)$.

Prove $F \cup G = F \cup (F^c \cap G) = F \cup (G - F)$ (A.16).
$F \cup G = (F \cup G) \cap \Omega$ by A.7.
$(F \cup G) \cap \Omega = (F \cup G) \cap (F \cup F^c)$ by A.10.
$(F \cup G) \cap (F \cup F^c) = F \cup (G \cap F^c)$ by A.17.
$\therefore F \cup (G \cap F^c) = F \cup (F^c \cap G)$ by A.8.
$G - F \triangleq G \cap F^c$
$\therefore F \cup (G \cap F^c) = F \cup (G - F)$
$\therefore F \cup G = F \cup (F^c \cap G) = F \cup (G - F)$

***A.4 Show that $F \subset G$ implies that $F \cap G = F$, $F \cup G = G$, and $G^c \subset F^c$.***
$F \subset G \Rightarrow F \cap G = F$
$F \subset G$ means that $\omega \in F \Rightarrow \omega \in G$.
$\omega \in F \cap G \Leftrightarrow \omega \in F$ and $\omega \in G \Rightarrow \omega \in F$
$\Rightarrow F \cap G \subset F$ and $F \subset G \cap F$
$\Rightarrow F \cap G = F$

$F \subset G \Rightarrow F \cup G = G$
$\omega \in F \cup G \Rightarrow \omega \in F$ or $\omega \in G$
$\Rightarrow \omega \in G$ or $\omega \in G$ because $F \subset G$.
$\Rightarrow \omega \in G$
$\Rightarrow F \cup G \subset G$.
$\omega \in G \Rightarrow \omega \in G \cap \Omega$
$\Rightarrow \omega \in G \cap (F \cup F^c)$
$\Rightarrow \omega \in (G \cap F) \cup (G \cap F^c)$
$\Rightarrow \omega \in F$ or $\omega \in (G \cap F^c)$
$\Rightarrow \omega \in F$ or $\omega \in G$
$\Rightarrow \omega \in F \cup G$
$\Rightarrow G \subset F \cup G$
$\Rightarrow F \cup G = G$

$F \subset G \Rightarrow G^c \subset F^c$
$\omega \in G^c \Leftrightarrow \omega \notin G$
$\Rightarrow \omega \notin F$
$\Rightarrow \omega \in F^c$
$\Rightarrow G^c \subset F^c$

***A.8 Prove the countably infinite version of deMorgan's "laws." For example, given a sequence of sets $F_i$; $i =$ 1, 2, . . . , then***

$$\bigcap_{i=1}^{\infty} F_i = \left( \bigcup_{i=1}^{\infty} F_i^c \right)^c .$$

To do this, we start by proving two subset relationships
$\omega \in \bigcap_{i=1}^{\infty} F_i$
$\Rightarrow \omega \in F_i$ for all $i$.
$\Rightarrow \omega \notin F_i^c$ for any $i$.
$\Rightarrow \omega \notin \bigcup_{i=1}^{\infty} F_i^c$
$\Rightarrow \omega \in (\bigcup_{i=1}^{\infty} F_i^c)^c$
$\therefore \bigcap_{i=1}^{\infty} F_i \subset (\bigcup_{i=1}^{\infty} F_i^c)^c$

$\omega \in (\bigcup_{i=1}^{\infty} F_i^c)^c$
$\Rightarrow \omega \notin \bigcup_{i=1}^{\infty} F_i^c$
$\Rightarrow \omega \notin F_i^c$ for any $i$.
$\Rightarrow \omega \in F_i$ for all $i$.
$\Rightarrow \omega \in \bigcap_{i=1}^{\infty} F_i$
$\therefore (\bigcup_{i=1}^{\infty} F_i^c)^c \subset \bigcap_{i=1}^{\infty} F_i$
Because these two are subsets of each other,
$\bigcap_{i=1}^{\infty} F_i = (\bigcup_{i=1}^{\infty} F_i^c)^c$

***A.12 Show that inverse images preserve set theoretic operations, that is, given $f : \Omega \to A$ and sets $F$ and $G$ in $A$, then***

$$f^{-1}(F^c) = \left(f^{-1}(F)\right)^c ,$$

$$f^{-1}(F \cup G) = f^{-1}(F) \cup f^{-1}(G) ,$$

***and***

$$f^{-1}(F \cap G) = f^{-1}(F) \cap f^{-1}(G) .$$

***If $\{F_i,\ i \in \mathcal{I}\}$ is an indexed family of subsets of $A$ that partitions $A$, show that $\left\{f^{-1}(F_i),\ i \in \mathcal{I}\right\}$ is a partition of $\Omega$. Do images preserve set theoretic operations in general? (Prove that they do or provide a counterexample).***

For $f^{-1}(F^c) = \left(f^{-1}(F)\right)^c$,
$\omega \in f^{-1}(F^c) \Leftrightarrow f(\omega) \in F^c \Leftrightarrow f(\omega) \notin F \Leftrightarrow \omega \notin f^{-1}(F) \Leftrightarrow \omega \in \left[f^{-1}(\omega)\right]^c$

For $f^{-1}(F \cup G) = f^{-1}(F) \cup f^{-1}(G)$,
$\omega \in f^{-1}(F \cup G) \Leftrightarrow f(\omega) \in F \cup G \Leftrightarrow f(\omega) \in F$ or $f(\omega) \in G$
$\Leftrightarrow \omega \in f^{-1}(F)$ or $\omega \in f^{-1}(G) \Leftrightarrow \omega \in f^{-1}(F) \cup f^{-1}(G)$

For $f^{-1}(F \cap G) = f^{-1}(F) \cap f^{-1}(G)$,
$\omega \in f^{-1}(F \cap G) \Leftrightarrow f(\omega) \in F \cap G \Leftrightarrow f(\omega) \in F$ and $f(\omega) \in G$
$\Leftrightarrow \omega \in f^{-1}(F)$ and $\omega \in f^{-1}(G) \Leftrightarrow \omega \in f^{-1}(F) \cap f^{-1}(G)$

Take $\{F_i,\ i \in \mathcal{I}\}$. If it is an indexed family of subests of $A$ that partitions $A$, this means that
$F_i \cap F_j = \emptyset$; all $i, j \in \mathcal{I},\ i \neq j$
and that
$\bigcup_{i \in \mathcal{I}} F_i = A$
We now need to show the same for the inverse image $\left\{f^{-1}(F_i),\ i \in \mathcal{I}\right\}$.
Our proofs above show that set theoretic operations are preserved for inverse images.
$f^{-1}(F_i) \cap f^{-1}(F_j) = f^{-1}(\emptyset) = \emptyset$; all $i, j \in \mathcal{I},\ i \neq j$
We now need to show that $\bigcup_{i \in \mathcal{I}} f^{-1}(F_i) = \Omega$.
This is true because $\bigcup_{i \in \mathcal{I}} f^{-1}(F_i) = f^{-1}\left(\bigcup_{i \in \mathcal{I}} F_i\right) = f^{-1}(A) = \Omega$.

Images do not preserve set theoretic operations in general. This is particularly well-illustrated for the case of non one-to-one mappings.
Let $\Omega = \{a,\, b,\, c\}$, $A = \{d,\, e\}$ with $f\,(a) = f\,(b) = d$ and $f\,(c) = e$.
Let $F = \{a\}$, $F^c = \{b,\, c\}$.
$f\,(F) = \{d\}$
$f\,(F^c) = f\,(\{b,\, c\}) = \{d,\, e\} \neq [f\,(F)]^c = \{e\}$

***2.3 Describe the sigma-field of subsets of $\Re$ generated by the points or singleton sets. Does this sigma-field contain interals of the form $(a,\, b)$ for $b > a$?***

The sigma-field $\mathcal{S}$ generated by the points must have all countable unions of distinct points of the form $\cup_i \{a_i\}$ together with the complements of such sets of the form $(\cup_i \{b_i\})^c = \cap_i \{b_i\}^c$, which are intersections of the sample space minus an individual point. Since $\mathcal{S}$ is a field, it must contain simple unions of the form

$$F = \cup_i \{a_i\} \cup \cap_j \{b_j\}^c .$$

The sigma-field does not contain intervals since intervals do not have the form of $F$.

***2.7 Let $\Omega = [0,\, \infty)$ be a sample space and let $\mathcal{F}$ be the sigma-field of subsets of $\Omega$ generated by all sets of the form $(n,\, n+1)$ for $n =$0, 1, 2, ...***
***(a) Are the following subsets of $\Omega$ in $\mathcal{F}$? (i) $[0,\, \infty)$, (ii) $\mathcal{Z}_+ = \{0,\, 1,\, 2, \ldots\}$, (iii) $[0,\, k] \cup [k+1,\, \infty)$ for any positive integer $k$, (iv) $\{k\}$ for any positive integer $k$, (v) $[0,\, k]$ for any positive integer $k$, (vi) $(1/3,\, 2)$.***
***(b) Define the following set function on subsets of $\Omega$:***

$$P\,(F) = c \sum_{i \in \mathcal{Z}_+ : i+1/2 \in F} 3^{-i}.$$

***(If there is no $i$ for which $i + 1/2 \in F$, then the sum is taken as zero.) Is $P$ a probability measure on $(\Omega, \mathcal{F})$ for an appropriate choice of $c$? If so, what is $c$?***
***(c) Repeat part (b) with $\mathcal{B}$, the Borel field, replacing $\mathcal{F}$ as the event space.***
***(d) Repeat part (b) with the power set of $[0,\, \infty)$ replacing $\mathcal{F}$ as the event space.***
***(e) Find $P\,(F)$ for the sets $F$ considered in part (a).***

(a) i) Yes, because $\Omega$ is always in $\mathcal{F}$.
ii) Yes, because this is the set that is formed by the complement of all of the subsets $(n,\, n+1)$ for all $n$= 0, 1, 2, ... This can be written

$$\mathcal{Z}_+ = \left( \bigcup_{n=0}^{\infty} (n,\, n+1) \right)^c \in \mathcal{F}$$

iii) $[0,\, k] \cup [k+1,\, \infty) = (k,\, k+1)^c \in \mathcal{F}$
iv) $\{k\} \notin \mathcal{F}$ since the set cannot be constructed as a countable combination of set theoretic operations on generating sets.
v) $[0,\, k] \notin \mathcal{F}$ since the set cannot be constructed as a countable combination of set theoretic operations on generating sets.
vi) $(1/3,\, 2)$ since the set cannot be constructed as a countable combination of set theoretic operations on generating sets.
b) This is a suitable probability measure if $P(\Omega) = 1$. It also satisifies the properties of nonnegativity and countable additivity.

$$P(\Omega) = 1 = c \sum_{k=0}^{\infty} 3^{-i} = \frac{c}{1 - 1/3} = \frac{3c}{2}$$

This means that $c = 2/3$.
c) This is going to be the same as in part (b), so $P$ is a valid probability measure with $c = 2/3$.
d) This is going to be the same as in part (b) since $P$ was defined for all sets and is thus a probability measure on the power set.

e) This problem is somewhat problematic because we shouldn't really take probability measures of sets that are not elements of the sigma-field.
i) $\Omega = [0,\ \infty)$ and thus $P(F) = 1$.
ii) $P(\mathcal{Z}_+) = 0$ as there are no $i$ for which $i + 1/2 \in \mathcal{Z}_+$.
iii) $P([0,\ k] \cup [k+1,\ \infty)) = P((k,\ k+1)^c) = 1 - P((k,\ k+1)) = 1 - \frac{2}{3}\left(3^{-k}\right)$.
iv) $P(\{k\}) = 0$ as there are no $k$ for which $k + 1/2 \in \mathcal{Z}_+$.
v) $P([0,\ k]) = \sum_{i=0}^{k-1} c3^{-i} = 1 - 3^{-(k)}$
vi) $P((1/3,\ 2)) = c\left(3^{-0} + 3^{-1}\right) = \frac{2}{3}\left(1 + \frac{1}{3}\right) = \frac{8}{9}$.

***2.9 Consider the measurable space*** $([0,\ 1],\ \mathcal{B}([0,\ 1]))$ ***.Define a set function*** $P$ ***on this space as follows:***

$$P(F) = \begin{cases} 1/2 & \text{if } 0 \in F \text{ or } 1 \in F \text{ but not both} \\ 1 & \text{if } 0 \in F \text{ and } 1 \in F \\ 0 & \text{otherwise} \end{cases}$$

***Is*** $P$ ***a probability measure?***
Yes. $P$ is a probability measure if it satisifes the three axioms for probability measures. It satisfies the property of nonnegativity and the property $P(\Omega) = 1$. We need to demonstrate countable additivity:
(a) $0 \notin F_i$ and $0 \notin F_i$ for all $i$. Then $P(\cup_i F_i) = 0 = \sum_i P(F_i)$.
(b) $0 \in F_i$ for some $i$ and $0 \notin F_i$ for all $i$, or $1 \in F_i$ for some $i$ and $1 \notin F_i$ for all $i$. Then $P(\cup_i F_i) = 1/2 = \sum_i P(F_i)$.
(c) $0 \in F_i$ for some $i$ and $0 \in F_j$ for some $j \neq k$. Then $P(\cup_i F_i) = 1 = \sum_i P(F_i)$.
(d) $0 \in F_i$ and $0 \in F_k$ for some $k$. Then $P(\cup_i F_i) = 1 = \sum_i P(F_i)$.
Thus $P$ is a probability measure.

***2.10 Let*** $\mathcal{S}$ ***be a sphere in*** $\Re^3$***:*** $\mathcal{S} = \left\{(x,\ y,\ z):\ x^2 + y^2 + z^2 \leq r^2\right\}$***, where*** $r$ ***is a fixed radius. In the sphere are fixed*** $N$ ***molecules of gas, each molecule being considered as an infinitesimal volume (that is, it occupies only a point in space). Define for any subset*** $F$ ***of*** $\mathcal{S}$ ***the function***

$$\#(F) = \{\text{the number of molecules in } F\}$$

***Show that*** $P(F) = \#(F)/N$ ***is a probability measure on the measurable space consisting of*** $\mathcal{S}$ ***and its power set.***
We need to demonstrate that this measure satifies the three axioms for probability measures.
$\#(F) \geq 0 \Rightarrow P(F) = P(F) = \#(F)/N \geq 0$ Nonnegativity
$\#(\mathcal{S}) = N \Rightarrow P(\Omega) = N/N = 1$ Normalization
Now we need to prove countable additivity.
For disjoint sets described by $\{F_i;\ i = 0,\ 1,\ \ldots,\ k-1\}$, we can say that any particle in $F_i$ is not in $F_j$ for $i \neq j$. Then $\#\left(\bigcup_{i=0}^{k-1} F_i\right) = \sum_{i=0}^{k-1} \#(F_i)$ and this implies $P\left(\bigcup_{i=0}^{k-1} F_i\right) = \sum_{i=0}^{k-1} P(F_i)$
Suppose now that the disjoint sets are a countable collection $\{F_i;\ i = 0,\ 1,\ \ldots\}$, let $M$ be the largest integer $i$ such that $\#(F) > 0$ (there must be such a finite integer since there are only $N$ particles). Then $\#\left(\bigcup_{i=M+1}^{\infty} F_i\right) = 0$ and

$$\begin{aligned} \#\left(\bigcup_{i=0}^{\infty} F_i\right) &= \#\left(\bigcup_{i=0}^{M} F_i\right) + \#\left(\bigcup_{i=M+1}^{\infty} F_i\right) \\ &= \#\left(\bigcup_{i=0}^{M} F_i\right) \\ &= \sum_{i=0}^{M} \#(F_i) = \sum_{i=0}^{\infty} \#(F_i) \end{aligned}$$

This implies that $P\left(\bigcup_{i=0}^{\infty} F_i\right) = \sum_{i=0}^{\infty} P(F_i)$ and hence $P$ is a probability measure.

***2.16 Prove that*** $P(F \cup G) \leq P(F) + P(G)$***. Prove more generally that for any sequence (i.e., countable collection) of events*** $F_i$***,***

$$P\left(\bigcup_{i=1}^{\infty} F_i\right) \leq \sum_{i=1}^{\infty} P(F_i).$$

***This inequality is called the union bound or the Bonferroni inequality. (Hint: use Problem A.2 or Problem 2.1).***

We know from 2.1 that in general, $P(F \cup G) = P(F) + P(G) - P(F \cap G)$

From non-negativity, we know that $P(F \cap G) \geq 0$ and thus $P(F \cup G) \leq P(F) + P(G)$.

Let $G_i = F_i - \bigcup_{j<i} F_j$ which makes these sets of $G$ disjoint.

$$\begin{aligned} P\left(\bigcup_i F_i\right) &= P\left(\bigcup_i G_i\right) \\ &= \sum_i P(G_i) \\ &= \sum_i P\left(F_i - \bigcup_{j<i} F_j\right) \\ &\text{We know that } P\left(F_i - \bigcup_{j<i} F_j\right) \leq P(F_i) \\ &\leq \sum_i P(F_i) \end{aligned}$$

***2.23 Answer true or false for each of the following statements. Answers must be justified.***
***(a) The following is a valid probability measure on the sample space $\Omega = \{1,2,3,4,5,6\}$ with event space $\mathcal{F} =$ all subsets of $\Omega$.***

$$P(F) = \frac{1}{21} \sum_{i \in F} i; \text{ all } F \in \mathcal{F}$$

True.

To prove this, we have to show that the probability measure satisifies the different axioms.

$P(F) \geq 0$ so nonnegativity is satisifed.

$P(\Omega) = 1$ so normalization is satisfied.

Now countable additivity needs to be proved.

If $F$ and $G$ are disjoint, then $P(F \cup G) = P(F) + P(G)$

$P(F) = \frac{1}{21} \sum_{\substack{i \in F \\ i \notin G}} i$

$P(G) = \frac{1}{21} \sum_{\substack{i \notin F \\ i \in G}} i$

$$P(F \cup G) = \frac{1}{21} \sum_{i \in (F \cup G)} i = \frac{1}{21} \left( \sum_{\substack{i \in F \\ i \notin G}} i + \sum_{\substack{i \notin F \\ i \in G}} i \right) = P(F) + P(G)$$

***(b) The following is a valid probaility measure on the sample space $\Omega = \{1,2,3,4,5,6\}$ with event space $\mathcal{F} =$ all subsets of $\Omega$:***

$$P(F) = \begin{cases} 1 & \text{if } 2 \in F \text{ or } 6 \in F \\ 0 & \text{otherwise} \end{cases}$$

False.

This is not a valid probability measure because countable additivity is not satisfied.

$P(\{2\}) = 1.$

$P(\{6\}) = 1.$

$P(\{2,6\}) = 1.$

$P(\{2,6\}) \neq P(\{2\}) + P(\{6\})$

***(c) If $P(G \cup F) = P(F) + P(G)$, then $F$ and $G$ are independent.***

False.

If $F$ and $G$ are independent, then $P(F \cap G) = P(F)P(G)$

By definition, $P(G \cup F) = P(F) + P(G) - P(F \cap G)$.

Because $P(F) > 0$ and $P(G) > 0$, then if $F$ and $G$ are independent, $P(F \cap G) > 0$ and $P(G \cup F) \neq P(F) + P(G)$

***(d)* $P(F|G) \geq P(G)$ *for all events* $F$ *and* $G$.**
False.
Just pick two disjoint events $F$ and $G$ with nonzero probability for $P(G)$.
Then, $P(F|G) = 0$ and $P(G) > 0$.

***(e) Muturally exclusive (disjoint) events with nonzero probability cannot be independent.***
True.
Suppose that $F$ and $G$ have nonzero probability so that $P(F)P(G) > 0$. Since the events are disjoint, $P(F \cap G) = 0$ and thus $P(F|G) = P(F \cap G)/P(G) = 0 \neq P(F)$. Thus, the events cannot be independent.

***(f) For any finite collection of events*** $F_i; i = 1, 2, \ldots, N$

$$P\left(\cup_{i=1}^{N} F_i\right) \leq \sum_{i=1}^{N} P(F_i)$$

True.
Define $G_n = F_n - \cup_{j<i} F_j$. Then $G_n \subset F_n$and the $G_n$ are disjoint so that

$$P\left(\cup_{i=1}^{N} F_i\right) = P\left(\cup_{i=1}^{N} G_i\right) = \sum_{i=1}^{N} P(G_i) \leq \sum_{i=1}^{N} P(F_i)$$

***2.26 Given a sample space*** $\Omega = \{0, 1, 2, \ldots\}$ ***define***

$$p(k) = \frac{\gamma}{2^k}; \ \ k = 0, 1, 2, \ldots$$

***(a) What must*** $\gamma$ ***be in order for*** $p(k)$ ***to be a pmf?***
To be a valid pmf, it must be positive for all values of $k$ and satisfy $\sum_{k=0}^{\infty} p(k) = 1$.
The infinite sum of a geometric progression with ratio $a$, $|a| < 1$ is

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}.$$

Thus, we can write:
$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{1-1/2} = 2 = \frac{1}{\gamma}$
$\gamma = \frac{1}{2}$ and our pmf is $p(k) = \frac{1}{2^{k+1}}$.

***(b) Find the probabilities*** $P(\{0, 2, 4, 6, \ldots\})$, $P(\{1, 3, 5, 7, \ldots\})$, ***and*** $P(\{1, 2, 3, 4, \ldots, 20\})$.
Even outcomes:
$P(\{0, 2, 4, 6, \ldots\}) = p(0) + p(2) + p(4) + \cdots = \sum_{i=0}^{\infty} p(2i) = \sum_{i=0}^{\infty} \frac{1}{2^{2i+1}}$
$= \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{2^{2i}} = \frac{1}{2} \sum_{i=0}^{\infty} \left(\frac{1}{4}\right)^i = \frac{1}{2} \cdot \frac{1}{1-1/4} = \frac{2}{3}$.
Odd outcomes:
$P(\{1, 3, 5, 6, \ldots\}) = 1 - P(\{0, 2, 4, 5, \ldots\}) = 1 - \frac{2}{3} = \frac{1}{3}$.
Finite outcomes:
Formula for finite sum of $N+1$ successive terms of geometric progression with ratio $a$:

$$\sum_{k=n}^{N+n} a^k = a^n \cdot \frac{1 - a^{N+1}}{1-a}$$

$P(\{1, 2, 3, 4, \ldots, 20\}) = \sum_{k=0}^{20} p(k) = \sum_{k=0}^{20} \frac{1}{2^{k+1}} = \frac{1}{2} \sum_{k=0}^{20} \left(\frac{1}{2}\right)^k = \frac{1}{2} \cdot \frac{1-(1/2)^{21}}{1-1/2} = 1 - \left(\frac{1}{2}\right)^{21} \approx 1 - 4.8 \times 10^{-7}$

***(c) Suppose that $K$ is a fixed integer. Find $P(\{0, K, 2K, 3K, \dots\})$.***
This is very similar to computing the even outcomes case above:
$P(\{0, K, 2K, 3K, \dots\}) = p(0) + p(K) + p(2K) + \dots = \sum_{i=0}^{\infty} p(Ki) = \sum_{i=0}^{\infty} \frac{1}{2^{Ki+1}}$
$= \frac{1}{2}\sum_{i=0}^{\infty} \frac{1}{2^{Ki}} = \frac{1}{2}\sum_{i=0}^{\infty} \left(\frac{1}{2^K}\right)^i = \frac{1}{2} \cdot \frac{1}{1-(1/2)^K} = \frac{2^{K-1}}{2^K-1}$.

***(d) Find the mean, second moment, and variance of this pmf.***
We know from a geometric pmf that $p(k) = (1-p)^{k-1}p$; k=1, 2, . . . . , where $p \in (0, 1)$ is a parameter that the mean is $1/p$ and the variance is $(1-p)/p^2$. Suppose that $p = 1/2$. This means that $p(1/2) = (1/2)(1/2)^{k-1} = (1/2)^k$. This is useful because this define the following sums:
$m = \sum_{k=0}^{\infty} k\left(\frac{1}{2}\right)^k = 1/p = 2$
$\sigma^2 = \sum_{k=0}^{\infty} (k-m)^2 \left(\frac{1}{2}\right)^k = \frac{1-p}{p^2} = \frac{1/2}{1/4} = 2$
$m^{(2)} = \sigma^2 + m^2 = 2 + 4 = 6 = \sum_{k=0}^{\infty} k^2 \left(\frac{1}{2}\right)^k$
The pmf of this problem can then be considered in relation to the geometric pmf
$m = \sum_{k=0}^{\infty} kp(k) = \sum_{k=0}^{\infty} \frac{k}{2^{k+1}} = \sum_{k=0}^{\infty} \frac{1}{2} \cdot k\left(\frac{1}{2}\right)^k = \frac{1}{2} \cdot \sum_{k=0}^{\infty} k\left(\frac{1}{2}\right)^k = \frac{1}{2} \cdot 2 = 1$
$m^{(2)} = \sum_{k=0}^{\infty} k^2 p(k) = \sum_{k=0}^{\infty} k^2 \cdot \frac{1}{2^{k+1}} = \sum_{k=0}^{\infty} \frac{1}{2} \cdot \frac{k^2}{2^k} = \frac{1}{2} \cdot \sum_{k=0}^{\infty} \frac{k^2}{2^k} = \frac{1}{2} \cdot 6 = 3$
$\sigma^2 = m^{(2)} - m^2 = 3 - 1 = 2$