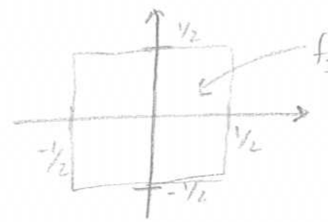
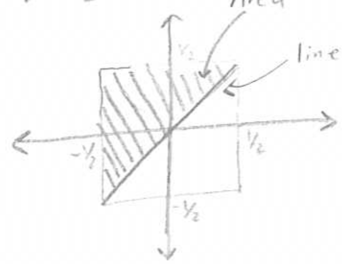
3.18) a)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} c\,dx\,dy = c = 1$$

$\therefore c = 1.$



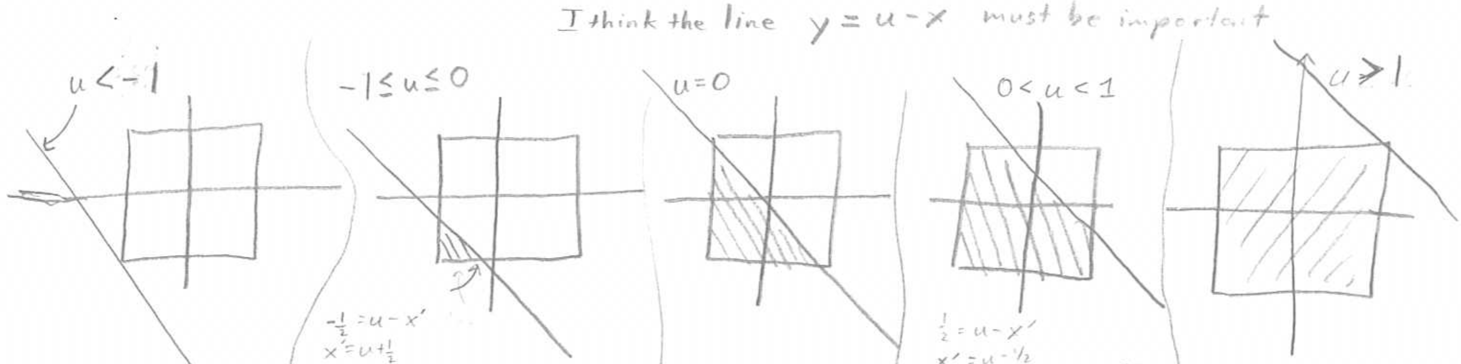

b) $P(\{x,y : x<y\}) = \iint_{Area} f_{X,Y}(x,y) = \frac{1}{2}(1)(1)c = \underline{1/2}$




c) $U(x,y) = x+y$

$F_U(u) = Pr(U \le u) = Pr(x+y \le u)$

I think the line $y = u - x$ must be important



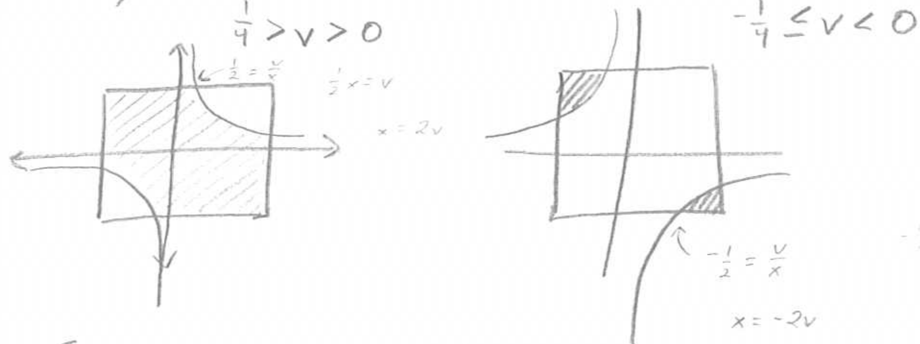

$P(U\le u) = 0$ | $P(U \le u) = \frac{(u+1)^2}{2}$ | $P(U\le u) = \frac{1}{2}$ | $P(U\le u) = 1 - \frac{(1-u)^2}{2}$ | $P(U \le u) = 1$

$$P(U\le u) = \begin{cases} 0, & u < -1 \\ \frac{(u+1)^2}{2}, & -1 \le u \le 0 \\ 1 - \frac{(1-u)^2}{2}, & 0 < u \le 1 \\ 1, & 1 < u \end{cases}$$

3.18 cont...)

d) $V: \mathbb{R}^2 \to \mathbb{R}$ $\quad V(x,y) = xy$ $\quad$ Find $F_V(v) = Pr(V \le v) = Pr(xy \le v)$

$y = \frac{v}{x}$ is the important equation




For $0 < v \le \frac{1}{4}$

$$F_V(v) = \frac{1}{4} + \frac{1}{4} + 2\left[\int_{2v}^{1/2} \frac{v}{x}\,dx + \int_0^{2v} \frac{1}{2}\,dx\right]$$

$$= \frac{1}{2} + 2\left[v + \left[v \ln x\right]_{2v}^{1/2}\right]$$

$$= \frac{1}{2} + 2v + 2v\left[\ln(\tfrac{1}{2}) - \ln(2v)\right]$$

$$= \frac{1}{2} + 2v - 2v\ln(4v)$$

For $-\frac{1}{4} \le v < 0$

$$F_V(v) = 2\int_{-2v}^{1/2} \left(\frac{v}{x} + \frac{1}{2}\right)dx = 2\left[v\ln x + \frac{1}{2}x\right]_{-2v}^{1/2}$$

$$= 2\left[v\ln(\tfrac{1}{2}) + \frac{1}{4} - v\ln(-2v) + v\right]$$

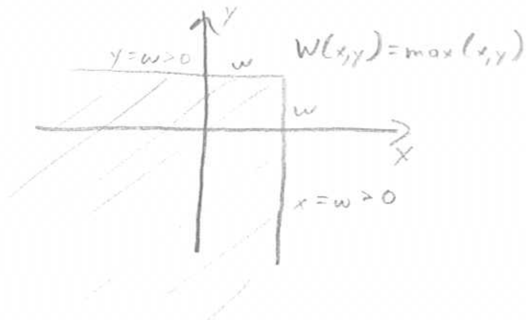
$$= \frac{1}{2} + 2v - 2v\ln(-4v)$$

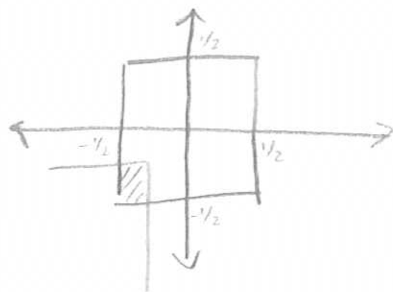
$$F_V(v) = \begin{cases} 0, & v \le -1/4 \\ \frac{1}{2} + 2v - 2v\ln(|4v|), & -1/4 < v < \frac{1}{4} \\ 1, & 1/4 \le v \end{cases}$$

3.18 cont...)

e) $W: \mathbb{R}^2 \to \mathbb{R}$

$W(x,y) = \max(x,y)$




$$F_W(w) = \begin{cases} 0, & w < -\frac{1}{2} \\ (w+\frac{1}{2})^2, & -\frac{1}{2} \le w \le \frac{1}{2} \\ 1, & \frac{1}{2} < w \end{cases}$$

3.20 cont...)

cont... a) Since $F_U(u) = \begin{cases} 0 & , u < 0 \\ 1 - (1+\lambda u)e^{-\lambda u} & , u \geq 0 \end{cases}$

$$f_U(u) = \frac{dF_U(u)}{du} = -\left[(1+\lambda u)(-\lambda)e^{-\lambda u} + \lambda e^{-\lambda u}\right]$$

$$= (\lambda + \lambda^2 u)e^{-\lambda u} - \lambda e^{-\lambda u}$$

$$= \underline{\underline{\lambda^2 u e^{-\lambda u}}}, \quad u \geq 0$$

b) $(W, V)$, $W = \min(X, Y)$, $V = \max(X, Y)$

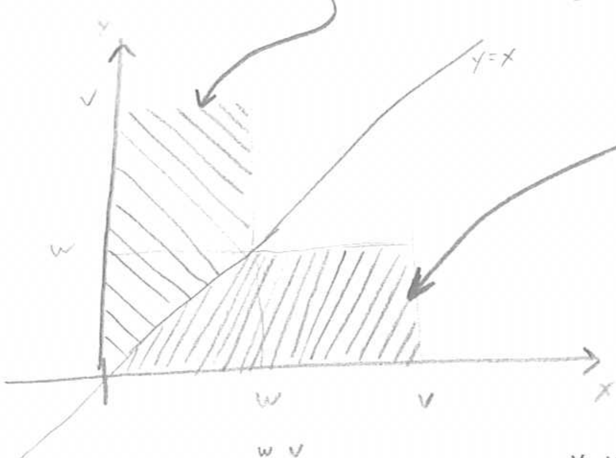
We have $W \leq V$. Thus, $f_{W,V}(w,v) = 0$ for $w > v$.

For $w \leq v$,

$$F_{W,V}(w,v) = P_r(W \leq w, V \leq v)$$

$$= P_r(\min(X,Y) \leq w, \max(X,Y) \leq v)$$

$$= P(Y \geq X, X \leq w, Y \leq v) + P(Y < X, Y \leq w, X \leq v)$$




$$F_{W,V}(w,v) = \int_0^w \int_0^v f_{X,Y}(x,y)\,dy\,dx + \int_w^v \int_0^w f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_0^w \int_0^v (\lambda e^{-\lambda x})(\lambda e^{-\lambda y})\,dy\,dx + \int_w^v \int_0^w (\lambda e^{-\lambda x})(\lambda e^{-\lambda y})\,dy\,dx$$

$$= \left[-e^{-\lambda y}\right]_0^v \left[-e^{-\lambda x}\right]_0^w + \left[-e^{-\lambda y}\right]_0^w \left[-e^{-\lambda x}\right]_w^v$$

$$= \left[1 - e^{-\lambda v}\right]\left[1 - e^{-\lambda w}\right] + \left[1 - e^{-\lambda w}\right]\left[e^{-\lambda w} - e^{-\lambda v}\right]$$

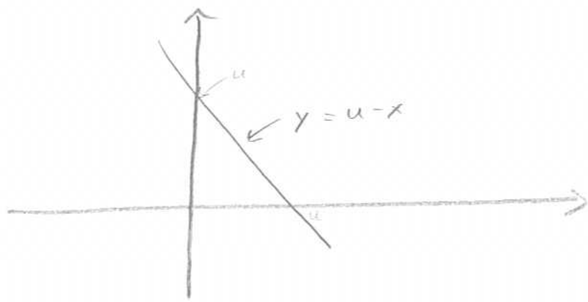
3.20)

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$$f_X(x) = f_Y(y) = \lambda e^{-\lambda x} \; ; \; x \geq 0$$

a) Find the pdf of $U = X + Y$.

Let's take a look at the CDF, $F_U(u) = P_r(U \leq u)$




$$P_r(U \leq u) = \int_0^u \int_0^{u-x} f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_0^u \int_0^{u-x} (\lambda e^{-\lambda x})(\lambda e^{-\lambda y})\,dy\,dx$$

$$= \int_0^u \lambda e^{-\lambda x} \left[ \int_0^{u-x} \lambda e^{-\lambda y}\,dy \right] dx$$

$$= \int_0^u \lambda e^{-\lambda x} \left[ -e^{-\lambda y} \right]_0^{u-x} dx$$

$$= \int_0^u \lambda e^{-\lambda x} \left[ -e^{-\lambda(u-x)} + 1 \right] dx$$

$$= \int_0^u \left( \left[ -\lambda e^{-\lambda x - \lambda u + \lambda x} \right] + \lambda e^{-\lambda x} \right) dx$$

$$= \int_0^u (-\lambda e^{-\lambda u} + \lambda e^{-\lambda x})\,dx$$

$$= \left[ -\lambda e^{-\lambda u} x \right]_0^u + \left[ -e^{-\lambda x} \right]_0^u$$

$$= -\lambda e^{-\lambda u} u + \left[ 1 - e^{-\lambda u} \right]$$

$$= 1 - (1 + \lambda u) e^{-\lambda u}$$

3.20 cont...)

b cont...)

$$F_{\underline{W},\underline{V}}(w,v) = \left(1 - e^{-\lambda v} - \cancel{e^{-\lambda w}} + e^{-\lambda(v+w)}\right) + \left(\cancel{e^{-\lambda w}} - e^{-\lambda v} - e^{-2\lambda w} + e^{-\lambda(v+w)}\right)$$

$$= 1 - 2e^{-\lambda v} - e^{-2\lambda w} + 2e^{-\lambda(v+w)}$$

$$f_{\underline{W},\underline{V}}(w,v) = \frac{\partial^2 F_{\underline{W},\underline{V}}(w,v)}{\partial v\,\partial w} = 2\lambda e^{-\lambda v} + 2(-\lambda)e^{-\lambda(v+w)}$$

$$\downarrow$$

$$= 2\lambda^2 e^{-\lambda(v+w)}$$

$$f_{\underline{W},\underline{V}}(w,v) = \begin{cases} 2\lambda^2 e^{-\lambda(v+w)} & , 0 \le w \le v \\ 0 & , \text{otherwise} \end{cases}$$

3.21) $(X, Y)$ with $P_{X,Y}$

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$$f_X(x) = f_Y(y) = \frac{e^{-r^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}$$

$$R = \sqrt{X^2 + Y^2}$$

$$\Theta = \tan^{-1}(Y/X)$$

Find joint pdf of $(R, \Theta)$

$$F_{R,\Theta}(r, \theta) = P_r(R \le r, \Theta \le \theta)$$

$$= \iint_{x,y:\sqrt{x^2+y^2} \le r,\ \tan^{-1}(y/x) \le \theta} f_{X,Y}(x,y)\, dx\, dy$$

$$= \iint_{x,y:\sqrt{x^2+y^2} \le r,\ \tan^{-1}(y/x) \le \theta} \frac{e^{-(x^2+y^2)/2\sigma^2}}{2\pi\sigma^2}\, dx\, dy$$

Change to polar coordinates

$$\rho^2 = x^2 + y^2 \qquad dx\,dy = \rho\, d\rho\, d\phi$$

$$\phi = \tan^{-1}(y/x)$$

$$F_{R,\Theta}(r, \theta) = \iint_{\rho,\phi:\rho \le r,\ -\pi \le \phi \le \theta} \frac{e^{-\rho^2/2\sigma^2}}{2\pi\sigma^2}\, \rho\, d\rho\, d\phi$$

$$= (\theta + \pi) \int_0^r \frac{e^{-\frac{\rho^2}{2\sigma^2}}}{2\pi\sigma^2}\, \rho\, d\rho \ ; \ \theta \in [-\pi, \pi],\ r \ge 0$$

$$f_{R,\Theta}(r, \theta) = \frac{\partial^2}{\partial r\, \partial\theta} F_{R,\Theta}(r, \theta)$$

$$= \frac{1}{2\pi} \frac{e^{-r^2/2\sigma^2}}{\sigma^2}\, r \ ; \ \theta \in [-\pi, \pi],\ r \ge 0$$

$$\Rightarrow f_\Theta(\theta) = \frac{1}{2\pi} \ ; \ \theta \in [-\pi, \pi] \qquad f_R(r) = \frac{e^{-r^2/2\sigma^2}}{\sigma^2}\, r \ ; \ r \ge 0$$

$\Theta$ and $R$ are independent.

3.22) $(\Omega, \mathcal{F}, P)$

$w = (w_0, \ldots, w_{k-1})$ where $w_i$ is 0 or 1.

$P$ is pmt with probability of $\frac{1}{2^8}$ to each of the $2^8$ elements.

a) $g(w) = \sum_{i=0}^{k-1} w_i$

Binomial with $n = 8$, $p = 0.5$

Random variable $G$ takes values in $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

$$P_G(g) = P(G = g) = P(\{w : \sum_{i=0}^{7} w_i = g\})$$

$$= \binom{8}{g} \cdot \frac{1}{2^8}$$

b) $X(w) = 1$ if there are an even number of $1^s$ in $w$ and $0$ otherwise.

$$P_X(1) = P(X = 1) = P(\{w : \sum_{i=0}^{7} w_i = 2k \text{ "even"}\})$$

$$= P_G(0) + P_G(2) + P_G(4) + P_G(6) + P_G(8)$$

$$= \frac{1}{2^8} \cdot \left(\binom{8}{0} + \binom{8}{2} + \binom{8}{4} + \binom{8}{6} + \binom{8}{8}\right)$$

$$= \frac{1}{2}$$

$$P_X(x) = \begin{cases} 1/2, & x = 0 \\ 1/2, & x = 1 \end{cases}$$

c) $Y(w) = w_j$, i.e. the value of the $j$th coordinate of $w$.

$$Y = \begin{cases} 1, & \text{if } w_j = 1 \\ 0, & \text{if } w_j = 0 \end{cases}$$

$$P_Y(1) = P(Y = 1) = P(\{w : w_j = 1\})$$

$$= \frac{(\text{\# outcomes such that } w_j = 1 \text{ and } w_i = 0 \text{ or } 1, i \neq j)}{\text{\# total outcomes}}$$

$$= \frac{\sum_{n=0}^{7} \binom{7}{n}}{2^8} = \frac{2^7}{2^8} = \frac{1}{2}$$

$$P_Y(y) = \begin{cases} 1/2, & y = 1 \\ 1/2, & y = 0 \end{cases}$$

d) $Z(\omega) = \max_i(\omega_i)$

$$Z = \begin{cases} 1, & \text{if } \max_i(\omega_i) = 1 \\ 0, & \text{if } \max_i(\omega_i) = 0 \end{cases}$$

$$p_Z(0) = P(Z=0) = P(\{\omega_i : \max_i(\omega_i) = 0\})$$
$$= P(\{0,0,0,0,0,0,0,0\}) = \frac{1}{2^8}$$

$$p_Z(1) = 1 - p_Z(0) = 1 - \frac{1}{2^8}$$

$$p_Z(z) = \begin{cases} 255/256, & z = 1 \\ 1/256, & z = 0 \end{cases}$$

e) $V(\omega) = g(\omega)X(\omega)$

$V$ can take on values $\{0, 2, 4, 6, 8\}$

$$p_V(2) = p_G(2) = \binom{8}{2} \cdot \frac{1}{2^8} = \frac{28}{256}$$

$$p_V(4) = p_G(4) = \binom{8}{4} \cdot \frac{1}{2^8} = \frac{70}{256}$$

$$p_V(6) = p_G(6) = \binom{8}{6} \cdot \frac{1}{2^8} = \frac{28}{256}$$

$$p_V(8) = p_G(8) = \binom{8}{8} \cdot \frac{1}{2^8} = \frac{1}{256}$$

$$p_V(0) = 1 - \left(p_V(2) + p_V(4) + p_V(6) + p_V(8)\right)$$

$$p_V(v) = \begin{cases} 129/256, & v = 0 \\ 28/256, & v = 2 \\ 70/256, & v = 4 \\ 28/256, & v = 6 \\ 1/256, & v = 8 \end{cases}$$

3.23) $(X_0, X_1, \ldots, X_N)$ is iid random vector with marginal pdf's

$$f_{X_n}(\alpha) = \begin{cases} 1 & 0 \le \alpha < 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{Uniform R.V.}$$

a) $U = X_0^2$

$$F_{X_0}(x) = \Pr(X_0 \le x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & 1 < x \end{cases}$$

$$F_U(u) = \Pr(U \le u) = \Pr(X_0^2 \le u) = \Pr(\sqrt{u} \le X_0 \le \sqrt{u})$$

$$= F_{X_0}(\sqrt{u}) = \begin{cases} 0, & u < 0 \\ \sqrt{u}, & 0 \le u \le 1 \\ 1, & 1 < u \end{cases}$$

$$f_U(u) = \frac{dF_U(u)}{du} = \begin{cases} \frac{1}{2}u^{-1/2}, & 0 \le u \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$V = \max(X_1, X_2, X_3, X_4)$ Range: $[0, 1]$

$$F_V(v) = \Pr(V \le v) = \Pr(\max(X_1, X_2, X_3, X_4) \le v)$$

$$= \Pr(X_1 \le v, X_2 \le v, X_3 \le v, X_4 \le v)$$

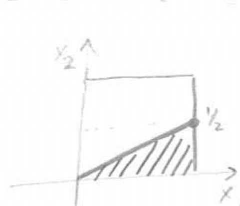
Because iid $= \Pr(X_1 \le v)\Pr(X_2 \le v)\Pr(X_3 \le v)\Pr(X_4 \le v)$

$$= F_{X_1}(v)F_{X_2}(v)F_{X_3}(v)F_{X_4}(v)$$

$$= \begin{cases} 0, & v < 0 \\ v^4, & 0 \le v \le 1 \\ 1, & 1 < v \end{cases}$$

$$f_V(v) = \begin{cases} 4v^3, & 0 \le v \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$W = \begin{cases} 1 & \text{if } X_1 \ge 2X_2 \\ 0 & \text{otherwise} \end{cases}$$




$$P(X_1 \ge 2X_2) = \iint_{\text{shaded area}} f_{X_1X_2}(x_1, x_2)\,dx_1\,dx_2 = 1/4$$

pmf: $p_W(w) = \begin{cases} 1/4, & \text{if } w = 1 \\ 3/4, & \text{if } w = 0 \end{cases}$

b) $Y_n = X_n + X_{n-1}$ ; $n = 1, \ldots, N$

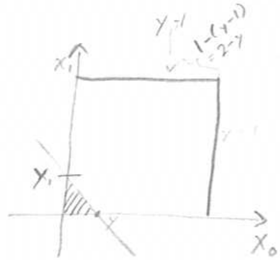
Range of $Y_n$ : $[0, N]$

$$F_{X_i}(x) = \Pr(X_i \leq x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq 1 \\ 1, & 1 < x \end{cases}$$

cdf of $Y_n$

$$F_{Y_n}(y) = \Pr(Y_n \leq y) = \Pr(X_n + X_{n-1} \leq y)$$

Since the $X_i$ are iid, this can be done for a representative $X_0$ and $X_1$.




$y = x_0 + x_1$

$x_1 = y - x_0$

$y = x_0 + 1$

$x_0 = y - 1$

$y = 1 + x_1$

This is thus exactly the same as 3.18 c) except shifted by ½

$$P(Y \leq y) = \begin{cases} 0, & y < 0 \\ \frac{1}{2} y^2, & 0 \leq y \leq 1 \\ 1 - \frac{1}{2}(2-y)^2, & 1 < y \leq 2 \\ 1, & 2 < y \end{cases}$$

3.35) $\vec{X} = (X_0, \dots, X_{k-1})$ is iid with marginal pmf

$$p_{X_i}(\ell) = p_X(\ell) = \begin{cases} p & \text{if } \ell = 1 \\ 1-p & \text{if } \ell = 0 \end{cases} \quad \text{for all } i.$$

a) Find pmf of $Y = \prod_{i=0}^{k-1} X_i$

$$p_Y(1) = P_r(\vec{X} = (1, \dots, 1))$$
$$= \prod_{i=0}^{k-1} p_{X_i}(1) \quad \text{because iid}$$
$$= p^k$$
$$p_Y(0) = 1 - p^k$$

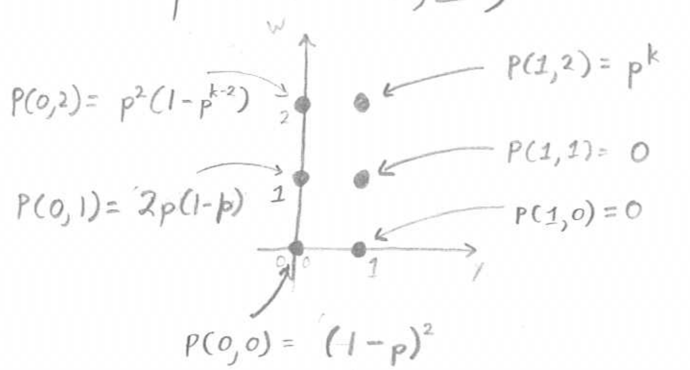
$$p_Y(y) = \begin{cases} 1-p^k, & \text{if } y = 0 \\ p^k, & \text{if } y = 1 \\ 0, & \text{otherwise} \end{cases}$$

b) $W = X_0 + X_{k-1}$

$W$ range is 0, 1, 2.

$$p_W(w) = \begin{cases} (1-p)^2, & \text{if } w = 0 \\ 2p(1-p), & \text{if } w = 1 \\ p^2, & \text{if } w = 2 \\ 0, & \text{otherwise} \end{cases}$$

c) pmf of $(Y, W)$




$\sum P(Y, W) =$

$p^k + p^2(1-p^{k-2}) + 2p(1-p) + (1-p)^2$

$= \cancel{p^k} + \cancel{p^2} - \cancel{p^k} + \cancel{2p} - \cancel{2p^2} + 1 - \cancel{2p} + \cancel{p^2}$

$= 1.$