

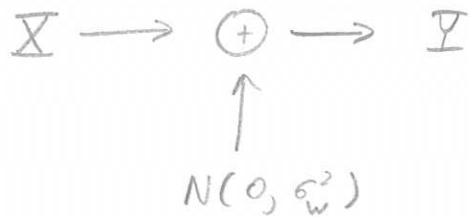
$$3.14) \quad \underline{X} \rightarrow \{a, b\}$$

$$P_{\underline{X}}(a) = p$$

$$P_{\underline{X}}(b) = 1-p$$

$$\underline{Y}: f_{\underline{Y}|\underline{X}}(y|x) = \frac{e^{-(y-x)^2/2\sigma_w^2}}{\sqrt{2\pi\sigma_w^2}}$$

We see that this is a situation where we have added Gaussian noise



We have  $f_{\underline{Y}|\underline{X}}(y|x)$ .

For our MAP detector we need  $f_{\underline{X}|\underline{Y}}(x|y)$ . Let's use Bayes' Rule.

Using (3.92)

$$\begin{aligned} P_{\underline{X}|\underline{Y}}(x|y) &= \frac{f_{\underline{Y}|\underline{X}}(y|x) P_{\underline{X}}(x)}{\sum_{\alpha} P_{\underline{X}}(\alpha) f_{\underline{Y}|\underline{X}}(y|\alpha)} = \frac{\left( \frac{e^{-(y-a)^2/2\sigma_w^2}}{\sqrt{2\pi\sigma_w^2}} \right) p \delta(x-a) + \left( \frac{e^{-(y-b)^2/2\sigma_w^2}}{\sqrt{2\pi\sigma_w^2}} \right) (1-p) \delta(x-b)}{\left[ \frac{e^{-(y-a)^2/2\sigma_w^2}}{\sqrt{2\pi\sigma_w^2}} + (1-p) \frac{e^{-(y-b)^2/2\sigma_w^2}}{\sqrt{2\pi\sigma_w^2}} \right]} \\ &= \frac{p e^{-(y-a)^2/2\sigma_w^2} \delta(x-a) + (1-p) e^{-(y-b)^2/2\sigma_w^2} \delta(x-b)}{p e^{-(y-a)^2/2\sigma_w^2} + (1-p) e^{-(y-b)^2/2\sigma_w^2}} \end{aligned}$$

$$P_{\underline{X}|\underline{Y}}(a|y) = \frac{p e^{-(y-a)^2/2\sigma_w^2}}{p e^{-(y-a)^2/2\sigma_w^2} + (1-p) e^{-(y-b)^2/2\sigma_w^2}}$$

$$P_{\underline{X}|\underline{Y}}(b|y) = \frac{(1-p) e^{-(y-b)^2/2\sigma_w^2}}{p e^{-(y-a)^2/2\sigma_w^2} + (1-p) e^{-(y-b)^2/2\sigma_w^2}}$$

3.14 cont...)

Our MAP detector (3.96) will be

$$\hat{X}(y) = \arg \max_{\mathbf{x}} P_{\mathbf{X}|\mathbf{I}}(\mathbf{x}|y)$$

$$= \arg \max_{\mathbf{x}} \left[ p e^{-\frac{(y-a)^2}{2\sigma_w^2}} \delta(x-a) + (1-p) e^{-\frac{(y-b)^2}{2\sigma_w^2}} \delta(x-b) \right]$$

We see there is a threshold when  $P_{\mathbf{X}|\mathbf{I}}(a|y) = P_{\mathbf{X}|\mathbf{I}}(b|y)$

$$p e^{-\frac{(y-a)^2}{2\sigma_w^2}} = (1-p) e^{-\frac{(y-b)^2}{2\sigma_w^2}}$$

$$\ln p - \frac{(y-a)^2}{2\sigma_w^2} = \ln(1-p) - \frac{(y-b)^2}{2\sigma_w^2}$$

$$2\sigma_w^2 (\ln p - \ln(1-p)) = (y-a)^2 - (y-b)^2$$

$$= y^2 - 2ay + a^2 - y^2 + 2by - b^2$$

$$= y(2b - 2a) + a^2 - b^2$$

$$y_{th} = \frac{2\sigma_w^2 (\ln p - \ln(1-p)) - a^2 + b^2}{2b - 2a}$$

This is the threshold where we decide between  $a$  and  $b$ .

$$P_e = \Pr(\hat{X}(\mathbf{I}) \neq X)$$

$$= \Pr(\hat{X}(\mathbf{I}) \neq a | X=a) P_X(a) + \Pr(\hat{X}(\mathbf{I}) \neq b | X=b) P_X(b)$$

let's say  $a \leq b$

$$= \Pr(Y > y_{th} | X=a) P_X(a) + \Pr(Y < y_{th} | X=b) P_X(b)$$

$$\left\{ \begin{array}{l} \Phi(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} du \\ Q(\alpha) = 1 - \Phi(\alpha) \end{array} \right.$$

$$= Q\left(\frac{y_{th}-a}{\sigma_w}\right) P_X(a) + Q\left(\frac{b-y_{th}}{\sigma_w}\right) P_X(b)$$

$$P_e = [Q\left(\frac{|y_{th}-a|}{\sigma_w}\right)]p + [Q\left(\frac{|b-y_{th}|}{\sigma_w}\right)](1-p)$$

3.14 cont...)

If  $p=0.5$ , this makes things much easier. Want to maximize distance between  $a$  and  $b$ . The threshold will be at  $\frac{a+b}{2}$ .

If  $(a^2 + b^2)/2 = E_b$ ,

$$\text{and } a = -b, \quad \frac{2a^2}{2} = E_b \rightarrow a = \pm\sqrt{E_b} \quad |a-b| = 2\sqrt{E_b}$$

If  $b=0$ ,

$$\frac{a^2}{2} = E_b \Rightarrow a = \sqrt{2E_b} \quad |a-b| = \sqrt{2E_b}$$

Thus, it is a much better choice under these constraints that  $a=-b$ .

We can see that the  $P_e$  will be minimized in this case and the threshold will be at zero. Thus

$$\begin{aligned} P_e &= \left[ Q\left(\frac{\sqrt{E_b}}{6w}\right) \right] (0.5) + \left[ Q\left(\frac{\sqrt{E_b}}{6w}\right) \right] (0.5) \\ &= Q\left(\frac{\sqrt{E_b}}{6w}\right) \end{aligned}$$

$$3.38) \quad \{X_n\}: P_X(1) = P_X(-1) = \frac{1}{2}$$

$$\{Y_n\}: N(0, 1) \stackrel{m=0}{\underset{\sigma=1}{\sim}}$$

$$\{Z_n = X_n + Y_n\}$$

a) Find pdf of  $Z_n$

$$\begin{aligned} F_{Z_n}(z) &= P(X+Y \leq z) \\ &= P(X_n + Y_n \leq z | X_n = +1) P_X(+1) + P(X_n + Y_n \leq z | X_n = -1) P_X(-1) \\ &= P(1 + Y_n \leq z) P_X(+1) + P(-1 + Y_n \leq z) P_X(-1) \\ &= P(Y_n \leq z-1) P_X(+1) + P(Y_n \leq z+1) P_X(-1) \\ &= \frac{1}{2} \int_{-\infty}^{z-1} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy + \frac{1}{2} \int_{-\infty}^{z+1} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \end{aligned}$$

$$\begin{aligned} f_{Z_n}(z) &= \frac{d}{dz} F_{Z_n}(z) \\ &= \frac{1}{2} \frac{1}{\sqrt{2\pi}} e^{-(z-1)^2/2} + \frac{1}{2} \frac{1}{\sqrt{2\pi}} e^{-(z+1)^2/2} \\ &= \frac{1}{2\sqrt{2\pi}} \left( e^{-(z^2-2z+1)/2} + e^{-(z^2+2z+1)/2} \right) \\ &= \frac{e^{-(z^2+1)/2}}{2\sqrt{2\pi}} (e^z + e^{-z}) \end{aligned}$$

b)  $\{R_n = \text{sgn}(Z_n)\}$

pmf of  $R_n$

We see that  $f_{Z_n}(z) = f_{Z_n}(-z)$ . Thus,  $P(Z < 0) = P(Z > 0) = \frac{1}{2}$

$$\Pr(R_n = 1) = \Pr(Z_n \geq 0) = \frac{1}{2}$$

$$\Pr(R_n = -1) = \Pr(Z_n < 0) = \frac{1}{2}$$

$$P_{R_n}(r) = \begin{cases} \frac{1}{2}, & r = 1 \\ \frac{1}{2}, & r = -1 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} \Pr(R_n = X_n) &= \Pr(R_n = X_n | X_n = 1) P_X(+1) + \Pr(R_n = X_n | X_n = -1) P_X(-1) \\ &= P(+1 + Y_n \geq 0) P_X(+1) + P(-1 + Y_n < 0) P_X(-1) \\ &= P(Y_n \geq -1) P_X(+1) + P(Y_n < 1) P_X(-1) \end{aligned}$$

3.38 cont...)

$$\Pr(R_n = \bar{X}_n) = P(Y_n < 1) (P_{\bar{X}}(+1) + P_{\bar{X}}(-1)) = P(Y_n < 1)$$

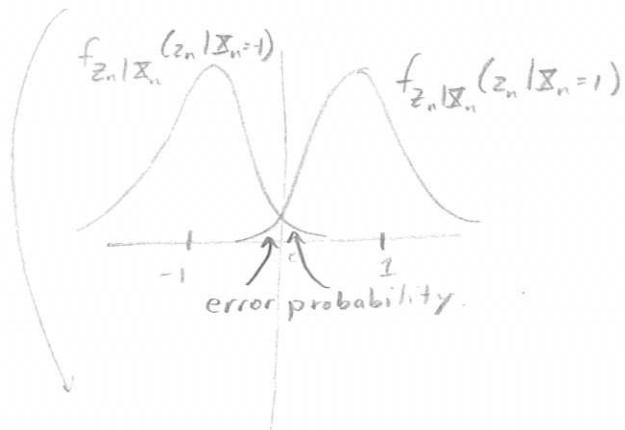
From standard normal table

$$P_c(R_n = \bar{X}_n) = 0.8413$$

c) Is this detector optimal?

If  $\hat{\bar{X}}_n$  is our approximation for  $\bar{X}_n$  then we want to minimize  $\Pr(\hat{\bar{X}}_n \neq \bar{X}_n)$  or maximize  $\Pr(\hat{\bar{X}}_n = \bar{X}_n)$ . The detector, to be optimal, needs  $\hat{\bar{X}}_n = 1$  if

$$\Pr(\bar{X}_n = 1 | Z_n = z_n) > \Pr(\bar{X}_n = -1 | Z_n = z_n)$$



$$\frac{1}{\sqrt{2\pi}} e^{-\frac{(z_n-1)^2}{2}} > \frac{1}{\sqrt{2\pi}} e^{-\frac{(z_n+1)^2}{2}}$$

$$(z_n-1)^2 < (z_n+1)^2$$

$$|z_n-1| < |z_n+1|$$

$$z_n > 0$$

Thus, this detector  $\{R_n = \text{sgn}(Z_n)\}$  is optimal.

$$3.39) \quad Y(t) = X \cos(2\pi f_0 t)$$

$$Y(t) = a(t)X$$

$$\begin{aligned} P(Y(t) \leq y) &= P(a(t)X \leq y) \\ &= P(X \leq \frac{y}{a(t)}) \end{aligned}$$

Since  $X$  is Gaussian

$$P(X \leq x) = \Phi\left(\frac{x-m}{\sigma}\right)$$

$$\begin{aligned} P(X \leq \frac{y}{a(t)}) &= \Phi\left(\frac{\frac{y}{a(t)} - m}{\sigma}\right) \\ &= \Phi\left(\frac{y - a(t)m}{a(t)\sigma}\right) \end{aligned}$$

$$N(a(t)m, [a(t)\sigma]^2)$$

So, the marginal pdf

$$f_{Y(t)} \text{ is } N(\cos(2\pi f_0 t)m, \cos^2(2\pi f_0 t)\sigma^2)$$

Where  $m$  is the mean of  $f_X(x)$  and  $\sigma^2$  is the variance of  $f_X(x)$ .

3.43)

Prove the following facts about characteristic functions:

a)  $|M_{\bar{X}}(ju)| \leq 1$

$$M_{\bar{X}}(ju) = E[e^{ju\bar{X}}] = \begin{cases} \sum_x p_{\bar{X}}(x) e^{jux} & \text{discrete} \\ \int f_{\bar{X}}(x) e^{jux} dx & \text{continuous} \end{cases}$$

$$|M_{\bar{X}}(ju)| = \begin{cases} \left| \sum_x p_{\bar{X}}(x) e^{jux} \right| \leq \left| \sum_x p_{\bar{X}}(x) \right| e^{|jux|} = \left| \sum_x p_{\bar{X}}(x) \right| = 1 \\ \left| \int f_{\bar{X}}(x) e^{jux} dx \right| \leq \left| \int f_{\bar{X}}(x) \right| e^{|jux|} = \left| \int f_{\bar{X}}(x) \right| = 1 \end{cases}$$

$$\therefore |M_{\bar{X}}(ju)| \leq 1$$

b)  $M_{\bar{X}}(0) = 1$

$$M_{\bar{X}}(0) = E[e^{0\cdot\bar{X}}] = E[1] = \begin{cases} \sum_x p_{\bar{X}}(x) & = 1 \\ \int f_{\bar{X}}(x) dx & \end{cases}$$

c)  $|M_{\bar{X}}(ju)| \leq M_{\bar{X}}(0) = 1$

From a)

$$|M_{\bar{X}}(ju)| \leq 1$$

From b)

$$M_{\bar{X}}(0) = 1$$

$$\therefore |M_{\bar{X}}(ju)| \leq M_{\bar{X}}(0) = 1$$

d)  $\bar{X}$  has characteristic function  $M_{\bar{X}}(ju)$

$$\bar{Y} = \bar{X} + c$$

$$\begin{aligned} M_{\bar{Y}}(ju) &= E[e^{ju\bar{Y}}] = E[e^{ju(\bar{X}+c)}] = \begin{cases} \sum_x p_{\bar{X}}(x) e^{ju(x+c)} \\ \int f_{\bar{X}}(x) e^{ju(x+c)} \end{cases} \\ &= \begin{cases} e^{juc} \sum_x p_{\bar{X}}(x) e^{jux} \\ e^{juc} \int f_{\bar{X}}(x) e^{jux} \end{cases} = e^{juc} E[e^{jux}] = e^{juc} M_{\bar{X}}(ju) \end{aligned}$$

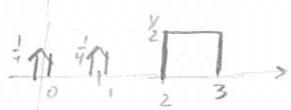
3.48) Two coins  $U$  is uniform over  $[0, 1]$ .

If first coin is "heads"  $p=0.5$

$$\underline{X} = \begin{cases} 1 & \text{if 2nd coin is "heads"} \\ 0 & \text{otherwise} \end{cases}$$

If first coin is "tails"  $p=0.5$ , then  $\underline{X}=U+2$

$$f_{\underline{X}}(x)$$



a)  $F_{\underline{X}}(x)$



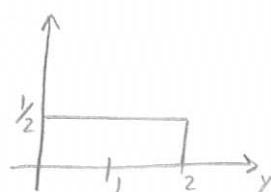
$$F_{\underline{X}}(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{4}, & 0 \leq x < 1 \\ \frac{1}{2}, & 1 \leq x < 2 \\ \frac{1}{2} + \frac{1}{2}(x-2), & 2 \leq x < 3 \\ 1, & 3 \leq x \end{cases}$$

b)  $\Pr(\frac{1}{2} \leq \underline{X} \leq 2) = F_{\underline{X}}(2) - F_{\underline{X}}(\frac{1}{2}) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

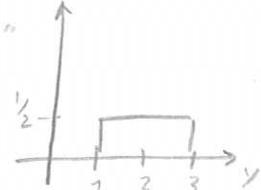
c)

$$\underline{Y} = \begin{cases} 2U & \text{if 1st coin is "heads"} \\ 2U+1 & \text{otherwise} \end{cases}$$

$$f_{\underline{Y}| \text{"Heads"}}$$

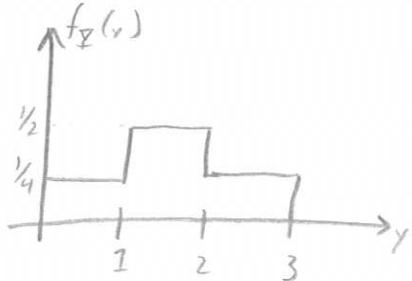


$$f_{\underline{Y}| \text{"tails"}}$$



$$f_{\underline{Y}}(y) = f_{\underline{Y}| \text{"heads"}}(y | \text{"heads"}) P(\text{"heads"}) + f_{\underline{Y}| \text{"tails"}}(y | \text{"tails"}) P(\text{"tails"})$$

$$f_{\underline{Y}}(y)$$



d) Design an optimal detection rule to estimate  $U$  if you are given only  $\bar{Y}$ . What is the probability of error?

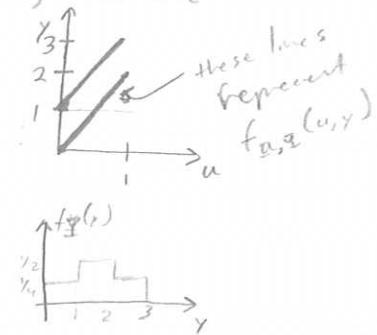
We propose an estimator  $\hat{U}(y)$ . We want to minimize  $\Pr(\hat{U} \neq U)$  or maximize  $\Pr(\hat{U} = U)$ .

$$\bar{Y} = \begin{cases} 2U & \text{if first coin is "heads"} \\ 2U+1 & \text{otherwise} \end{cases}$$

$$P_{\bar{Y}|U}(y|u) = \begin{cases} \frac{1}{2}, & \text{if } y = 2u \\ \frac{1}{2}, & \text{if } y = 2u+1 \end{cases} \quad f_U(u) = \begin{cases} 1, & \text{if } 0 \leq u \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{U|\bar{Y}}(u|y) = \frac{P_{\bar{Y}|U}(y|u) f_U(u)}{f_{\bar{Y}}(y)} = \frac{f_{U,\bar{Y}}(u,y)}{f_{\bar{Y}}(y)} =$$

$$f_{\bar{Y}|U}(u|y) = \begin{cases} 1 & \text{if } u = \frac{y}{2}, \quad 0 \leq y < 1 \\ 0 & \text{if } u \neq \frac{y}{2}, \quad 0 \leq y < 1 \\ \frac{1}{2} & \text{if } u = \frac{y-1}{2}, \quad 1 \leq y < 2 \\ \frac{1}{2} & \text{if } u = \frac{y-1}{2}, \quad 1 \leq y < 2 \\ 1 & \text{if } u = \frac{y-1}{2}, \quad 2 \leq y < 3 \\ 0 & \text{if } u \neq \frac{y-1}{2}, \quad 2 \leq y < 3 \end{cases}$$



$$\hat{U}(y) = \arg \max_u f_{\bar{Y}|U}(u|y)$$

$$= \begin{cases} \frac{y}{2}, & 0 \leq y < 1 \\ \frac{y-1}{2}, & 1 \leq y < 2 \\ \frac{y-1}{2}, & 2 \leq y < 3 \end{cases} \quad \text{or} \quad \text{equally optimal} = \begin{cases} \frac{y}{2}, & 0 \leq y < 1 \\ \frac{y-1}{2}, & 1 \leq y < 2 \\ \frac{y-1}{2}, & 2 \leq y < 3 \end{cases}$$

$$\begin{aligned} P_e &= \Pr(\hat{U} \neq U) = \Pr(\hat{U} \neq u | U=u) \\ &= \Pr\left(\frac{\bar{Y}}{2} \neq u | 1 \leq y < 2\right) \Pr(1 \leq y < 2) \\ &= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \end{aligned}$$

3.48 e))

$$P_Z(1) = p \quad , \text{ given } U$$

$$Z = \begin{cases} 1 & , 0 \leq U < p \\ 0 & , \text{ otherwise} \end{cases}$$

This is how computers frequently do this calculation.

ii) Generate a continuous, uniformly distributed variable given  $Z$ .

if we have  $p = 0.5$ , and an infinite family of  $Z_n$ , for  $n=0, 1, \dots$   
we can generate a uniform random variable

$$U = \sum_{n=0}^{\infty} b_n(r) 2^{-n+1}$$

$$\text{for } b_n(r) = Z_n$$

$$3.55) \quad \{X_n; n=0, 1, 2, \dots\} \text{ iid}$$

$$P_{X_n}(k) = \begin{cases} P & \text{if } k = 1 \\ 1-P & \text{if } k = 0 \end{cases} \quad \text{for all } n$$

$$\{\bar{W}_n; n=0,1,\dots\} \text{ iid}$$

$$P_{\overline{W}_n}(k) = \begin{cases} \epsilon & \text{if } k=1 \\ 1-\epsilon & \text{if } k=0 \end{cases}$$

$$\mathbb{F}_n = \mathbb{X}_n \oplus \mathbb{W}_n$$

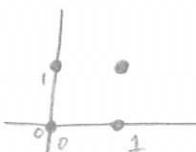
$$a) P_{I_n}(k) = \begin{cases} (1-p)(1-\epsilon) + p\epsilon & , \text{if } k=0 \\ p(1-\epsilon) + (1-p)\epsilon & , \text{if } k=1 \\ p^2p\epsilon + \epsilon^2 + p\epsilon & , \text{if } k=2 \end{cases}$$

$$\overline{Y}_n = 1 \text{ if } \overline{X}_n = 0, \quad \overline{W}_n = 1 \quad (1-p) \epsilon$$

$$\underline{X}_n = 1, \quad \underline{W}_n = 0 \quad p(1-\epsilon)$$

$$T_n = 0 \text{ if } X_n = 0, \quad T_n = 1 \text{ if } X_n = 1$$

$$X_n = 1, \quad W_n = 1 \quad p \in$$



b)  $\{I_n\}$  is Bernoulli, or i.i.d. because there is no memory in the system.

$$c) P_{\bar{X}_n | \bar{X}_n}(j|k) = \begin{cases} \epsilon & , j \neq k \\ 1-\epsilon & , j = k \end{cases}$$

$$d) P_{X_n | Y_n}(k|j) = \begin{cases} \frac{(1-p)(1-\epsilon)}{(1-p)(1-\epsilon)+p\epsilon} & \text{if } k=0, j=0 \\ \frac{p\epsilon}{(1-p)(1-\epsilon)+p\epsilon} & \text{if } k=1, j=0 \\ \frac{(1-p)\epsilon}{(1-p)\epsilon+p(1-\epsilon)} & \text{if } k=0, j=1 \\ \frac{p(1-\epsilon)}{(1-p)\epsilon+p(1-\epsilon)} & \text{if } k=1, j=1 \end{cases}$$

$$e) \Pr(Y_n \neq X_n) = \Pr(Y=1|X=0)\Pr(X=0) + \Pr(Y=0|X=1)\Pr(X=1) \\ = (\epsilon)(1-p) + (\epsilon)(p) = \epsilon$$

f) Estimate  $\hat{X}(j)$

$P_e = \Pr(\hat{X}(Y_n) \neq X_n)$ . Consider part d)  $P_{\hat{X}_n | Y_n}(k | j)$ .

If  $j=0$ , we decide  $\hat{X}(j)=0$  if  $(1-p)(1-\epsilon) > p\epsilon$ , otherwise  $\hat{X}(j)=1$ .  
 $1-p-\epsilon + p\epsilon > p\epsilon$   
 $1 > p+\epsilon$

If  $j=1$ , we decide  $\hat{X}(j)=1$  if  $p(1-\epsilon) > (1-p)\epsilon$ , otherwise  $\hat{X}(j)=0$ .  
 $p - p\epsilon > \epsilon - p\epsilon$   
 $p > \epsilon$

Using this decision rule,

$$\begin{aligned}
 P_e &= \Pr(\hat{X}(Y_n) \neq X_n) \\
 &= \Pr(\hat{X}(Y_n) = 1 | \bar{X} = 0) \Pr(\bar{X} = 0) + \Pr(\hat{X}(Y_n) = 0 | \bar{X} = 1) \Pr(\bar{X} = 1) \\
 &= \left[ \Pr(\hat{X}(Y_n) = 1 | \bar{X} = 0, Y = 0) + \Pr(\hat{X}(Y_n) = 1 | \bar{X} = 0, Y = 1) \right] \Pr(\bar{X} = 0) \\
 &\quad + \left[ \Pr(\hat{X}(Y_n) = 0 | \bar{X} = 1, Y = 0) + \Pr(\hat{X}(Y_n) = 0 | \bar{X} = 1, Y = 1) \right] \Pr(\bar{X} = 1) \\
 &= \Pr(\hat{X}(Y_n) = 1 | \bar{X} = 0, Y = 0) \Pr(\bar{X} = 0, Y = 0) \xrightarrow{(1-p)(1-\epsilon)} + \Pr(\hat{X}(Y_n) = 0 | \bar{X} = 0, Y = 1) \Pr(\bar{X} = 0, Y = 1) \xrightarrow{p(1-\epsilon)} \\
 &\quad + \Pr(\hat{X}(Y_n) = 0 | \bar{X} = 1, Y = 0) \Pr(\bar{X} = 1, Y = 0) \xrightarrow{p\epsilon} + \Pr(\hat{X}(Y_n) = 0 | \bar{X} = 1, Y = 1) \Pr(\bar{X} = 1, Y = 1) \xrightarrow{p(1-\epsilon)}
 \end{aligned}$$

If  $1 > p+\epsilon, p > \epsilon$ :  $P_e = p\epsilon + (1-p)\epsilon$

$1 > p+\epsilon, p \leq \epsilon$ :  $P_e = p\epsilon + p(1-\epsilon)$

$1 \leq p+\epsilon, p > \epsilon$ :  $P_e = (1-p)(1-\epsilon) + (1-p)\epsilon$

$1 \leq p+\epsilon, p \leq \epsilon$ :  $P_e = (1-p)(1-\epsilon) + p(1-\epsilon)$