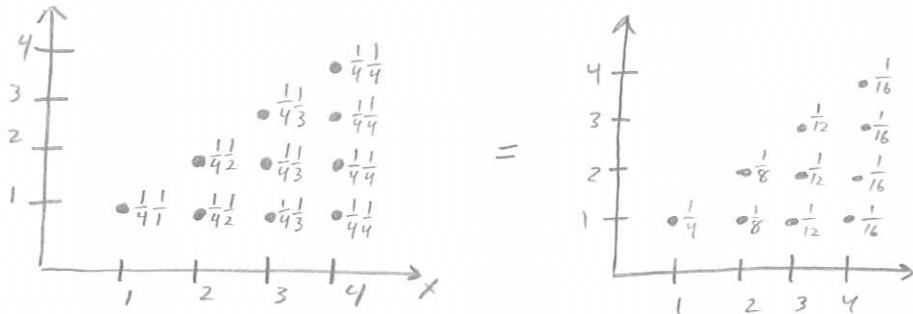


$$4.1) P_{\bar{X}}(k) = \begin{cases} \frac{1}{4^k}, & k=1, 2, 3, 4 \\ 0, & \text{otherwise} \end{cases}$$

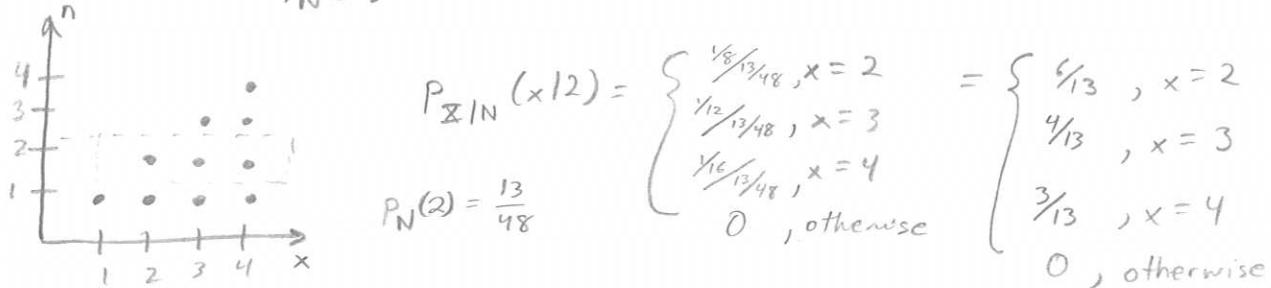
$$P_{N|\bar{X}}(n|k) = \frac{1}{k}; n=1, 2, \dots, k.$$

$$a) P_{\bar{X}, N}(x, n) = P_{N|\bar{X}}(n|x) P_{\bar{X}}(x)$$



$$b) P_N(n) = \begin{cases} \frac{1}{4} + \frac{1}{8} + \frac{1}{12} + \frac{1}{16}, & n=1 \\ \frac{1}{8} + \frac{1}{12} + \frac{1}{16}, & n=2 \\ \frac{1}{12} + \frac{1}{16}, & n=3 \\ \frac{1}{16}, & n=4 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{25}{48}, & n=1 \\ \frac{13}{48}, & n=2 \\ \frac{7}{48}, & n=3 \\ \frac{1}{16}, & n=4 \\ 0, & \text{otherwise} \end{cases}$$

$$c) P_{\bar{X}|N}(x|n) = \frac{P_{\bar{X}, N}(x, n)}{P_N(n)}$$



$$d) P_{\bar{X}|A} = \frac{P_{\bar{X}, A}(x)}{P(A)} = \frac{P_{\bar{X}, A}(x)}{\frac{1}{12} + \frac{1}{16} + \frac{1}{8} + \frac{1}{12} + \frac{1}{16}} = \frac{P_{\bar{X}, A}(x)}{\frac{4+3+6+4+3}{48}} = \frac{P_{\bar{X}, A}(x)}{\frac{20}{48}} = \frac{P_{\bar{X}, A}(x)}{\frac{5}{12}}$$

$$P_{\bar{X}|A}(x) = \begin{cases} \frac{1}{8}/\frac{5}{12}, & x=2 \\ \frac{1}{12} + \frac{1}{12}/\frac{5}{12}, & x=3 \\ \frac{1}{16} + \frac{1}{16}/\frac{5}{12}, & x=4 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{3}{10}, & x=2 \\ \frac{2}{5}, & x=3 \\ \frac{3}{10}, & x=4 \\ 0, & \text{otherwise} \end{cases}$$

$$e) E[C] = 3 \text{ where } C \text{ is cost of a book}$$

$$T = \sum_{i=1}^n C_i \quad \text{where } T \text{ is total cost and } n \text{ is books bought}$$

$$E[T] = E[N]E[C] = \frac{7}{4} \cdot 3 = \boxed{\frac{21}{4}}$$

$$\hookrightarrow E[N] = (1)\frac{25}{48} + (2)\left(\frac{13}{48}\right) + (3)\left(\frac{7}{48}\right) + (4)\left(\frac{1}{48}\right) = \frac{25}{48} + \frac{26}{48} + \frac{21}{48} + \frac{12}{48} = \frac{84}{48} = \frac{21}{12} = \frac{7}{4}$$

$$E[\bar{X}|A] = 2\left(\frac{3}{10}\right) + 3\left(\frac{2}{5}\right) + 4\left(\frac{3}{10}\right) = \frac{3}{5} + \frac{6}{5} + \frac{6}{5} = \frac{15}{5} = \boxed{3}$$

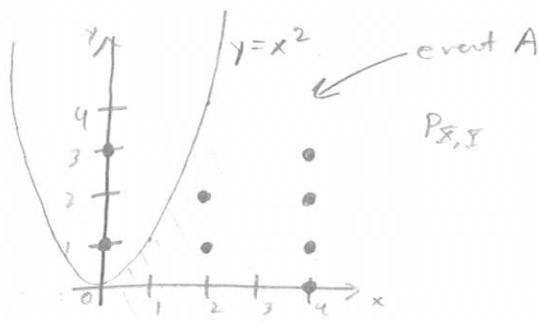
$$E[\bar{X}^2|A] = 4\left(\frac{3}{10}\right) + 9\left(\frac{2}{5}\right) + 16\left(\frac{3}{10}\right) = \frac{6}{5} + \frac{18}{5} + \frac{24}{5} = \frac{48}{5}$$

$$\text{Var}(\bar{X}^2|A) = E[\bar{X}^2|A] - (E[\bar{X}|A])^2 = \frac{48}{5} - 9 = \frac{48}{5} - \frac{45}{5} = \boxed{\frac{3}{5}}$$

q. 5. Given the setup of Problem 3.10:

$$a) E[\bar{X}] = \frac{1}{4}(0) + \frac{1}{4}(2) + \frac{1}{2}(4) = \frac{1}{2} + 2 = \frac{5}{2}$$

$$\begin{aligned} E[\bar{XY}] &= \frac{1}{8}(0+0+2+4+0+4+8+12) \\ &= \frac{1}{8}(30) = \frac{30}{8} = \frac{15}{4} \end{aligned}$$



$$b) E[\bar{Y} | \bar{X}=x] = \begin{cases} \frac{1}{2}(1+3), & x=0 \\ \frac{1}{2}(1+2), & x=2 \\ \frac{1}{4}(0+1+2+3), & x=4 \end{cases} = \begin{cases} 2, & x=0 \\ \frac{3}{2}, & x=2 \\ \frac{3}{2}, & x=4 \end{cases}$$

This is maximized for $x=0$.

c) A denotes event $\bar{X}^2 \geq 9$

$$P(A) = P(\{(2,1), (2,2), (4,0), (4,1), (4,2), (4,3)\}) = \frac{6}{8} = \frac{3}{4}$$

$$\begin{aligned} E(\bar{XY}|A) &= \sum_{x,y} xy P_{\bar{X}\bar{Y}|A} (xy|A) = \frac{1}{8} \left(2+4+0+4+8+12 \right) \\ &= \frac{1}{6}(30) = \boxed{5} \end{aligned}$$

4.6) X is R.V. with pdf $f_X(x)$ and c.f. $M_X(ju) = E[e^{juX}]$

$$Y = aX + b \quad a, b > 0$$

Find pdf f_Y and c.f. $M_Y(ju)$ in terms of f_X , M_X

$$F_X(x) = P(X \leq x) = \int_{-\infty}^x f_X(\alpha) d\alpha$$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(aX + b \leq y) = P(aX \leq y - b) = P(X \leq \frac{y-b}{a}) \\ &= F_X\left(\frac{y-b}{a}\right) = \int_{-\infty}^{\frac{y-b}{a}} f_X(\alpha) d\alpha \end{aligned}$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d}{dy} \int_{-\infty}^{\frac{y-b}{a}} f_X(\alpha) d\alpha = \boxed{\frac{1}{a} f_X\left(\frac{y-b}{a}\right)} \quad \text{Formula 3.41}$$

$$\begin{aligned} M_Y(ju) &= E[e^{juY}] = E[e^{ju(aX+b)}] = E[e^{juaX} e^{jub}] = e^{jub} E[e^{juaX}] \\ &= \underline{e^{jub} M_X(jua)} \end{aligned}$$

4.7) X, Y, Z are iid Gaussian with $N(1, 1)$

$$\bar{V} = 2X + Y$$

$$\bar{W} = 3X - 2Z + 5$$

$$\text{Var}(X) = 1 = E[X^2] - (E[X])^2 = E[X^2] - 1$$

$$E[X^2] = 2$$

$$a) E[\bar{V}\bar{W}] = E[(2X+Y)(3X-2Z+5)]$$

$$= E[6X^2 - 4XZ + 10X + 3XY - 2YZ + 5Y]$$

$$= 6E[X^2] - 4E[X]E[Z] + 10E[X] + 3E[X]E[Y] - 2E[Y]E[Z] + 5E[Y]$$

$$= 6(2) - 4 + 10 + 3 - 2 + 5$$

$$= 12 + 12$$

$$= 24$$

b) The two parameters that completely specify $\bar{V} + \bar{W}$ are the mean and variance because they are Gaussian.

$$\bar{V} + \bar{W} = 2X + Y + 3X - 2Z + 5$$

$$= 5X + Y - 2Z + 5$$

$$E[\bar{V} + \bar{W}] = E[5X + Y - 2Z + 5] = 5E[X] + E[Y] - 2E[Z] + 5$$

$$= 5 + 1 - 2 + 5 = 9$$

$$\text{Var}(\bar{V} + \bar{W}) = \text{Var}(5X + Y - 2Z + 5) = 5^2 \text{Var}(X) + \text{Var}(Y) + 4 \text{Var}(Z)$$

$$= 25 + 1 + 4 = 30$$

This is completely specified by mean of 9 and variance of 30.

c) Find c.f. of random vector $[\bar{V} \bar{W}]^t$

C.f. for Gaussian is

$$E[\bar{V}] = 2E[X] + E[Y] = 2 + 1 = 3$$

$$E[\bar{W}] = 3E[X] - 2E[Z] + 5 = 6$$

$$\text{Var}(\bar{V}) = 4\text{Var}(X) + \text{Var}(Y) = 5$$

$$\text{Var}(\bar{W}) = 9\text{Var}(X) + 4\text{Var}(Z) = 13$$

$$\text{cov}(\bar{V}, \bar{W}) = E[\bar{V}\bar{W}] - E[\bar{V}]E[\bar{W}] = 24 - (3)(6) = 6$$

This is a cascade vector with $R_{\bar{V}\bar{W}\bar{V}\bar{W}} = \begin{bmatrix} 5 & 6 \\ 6 & 13 \end{bmatrix}$ and $\vec{m} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$$H(j\hat{u}) = e^{j\hat{u}^t \vec{m} - \hat{u}^t \Lambda \hat{u}/2} \quad \text{where } \vec{m} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} 5 & 6 \\ 6 & 13 \end{bmatrix}$$

d) The linear estimator $\hat{V}(W)$ of V , given W is of the form $\hat{V}(W) = AW$

$$R_{\bar{W}} = E[\bar{W} \bar{W}^T] = E[\bar{W}^2] = \text{var}(W) + (E[\bar{W}])^2 = 13 + 6^2 = 13 + 36 = 49$$

$$A^T = R_{\bar{W}}^{-1} E[\bar{W} \bar{W}^T] = \frac{1}{49} \cdot 24$$

$$\hat{V}(W) = \frac{24}{49} \bar{W}$$

e) The optimal (smallest MSE) affine estimator is of form

$$\hat{V}(W) = aW + b$$

$$a = \frac{\text{cov}(V, W)}{\text{var}(W)} = \frac{6}{13}$$

$$b = E[V] - E[W] \frac{\text{cov}(V, W)}{\text{var}(W)} = 3 - 6 \left(\frac{6}{13} \right) = 3 - \frac{36}{13} = \frac{39 - 36}{13} = \frac{3}{13}$$

$$\hat{V}(W) = \frac{6}{13} W + \frac{3}{13}$$

f) This is the optimal estimator because V and W are both Gaussian

g) $\bar{X}-\bar{\bar{X}}, \bar{Y}-\bar{\bar{Y}}, \bar{Z}-\bar{\bar{Z}}$ inputs to black box

Two outputs A and B

$$\begin{bmatrix} A \\ B \end{bmatrix} \quad \text{You want } L_{AB} = \begin{bmatrix} 3 & 2 \\ 2 & 5 \end{bmatrix}$$

This means you have $\text{var}(A) = 3$ and $\text{var}(B) = 5$ and $\text{cov}(A, B) = 2$.

$$\begin{aligned} \text{You have } A &= a_1 \overset{\circ}{X} + a_2 \overset{\circ}{Y} + a_3 \overset{\circ}{Z} & \text{where } \overset{\circ}{X} = X - \bar{X} & N(0, 1) & E[\overset{\circ}{X}] = 0 \\ B &= b_1 \overset{\circ}{X} + b_2 \overset{\circ}{Y} + b_3 \overset{\circ}{Z} & \overset{\circ}{Y} = Y - \bar{Y} & N(0, 1) & E[\overset{\circ}{Y}]^2 = 1 \\ & & \overset{\circ}{Z} = Z - \bar{Z} & N(0, 1) & \end{aligned}$$

$$\begin{aligned} \text{cov}(A, B) = 2 &= E[AB] - E[A]E[B] = a_1 b_1 E[\overset{\circ}{X}^2] + a_2 b_2 E[\overset{\circ}{Y}^2] + a_3 b_3 E[\overset{\circ}{Z}^2] \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3 \end{aligned}$$

$$\text{var}(A) = 3 = a_1^2 + a_2^2 + a_3^2$$

$$\text{var}(B) = 5 = b_1^2 + b_2^2 + b_3^2$$

An example thus is: $a_1, b_1 = 2$

$$a_1 = 1, b_1 = 2$$

$$a_2 = \sqrt{2}, b_2 = 0$$

$$a_3 = 0, b_3 = 1$$

g cont...)

This would satisfy this

$$A = \overset{\circ}{X} + \sqrt{2} \overset{\circ}{Y}$$

$$B = 2\overset{\circ}{X} + \overset{\circ}{Z}$$

$$\text{cov}(A, B) = E[AB] - E[A]E[B] = E[2\overset{\circ}{X}^2] = 2E[\overset{\circ}{X}^2] = 2$$

$$\text{var}(A) = \text{var}(\overset{\circ}{X}) + 2\text{var}(\overset{\circ}{Y}) = 3$$

$$\text{var}(B) = 4\text{var}(\overset{\circ}{X}) + \text{var}(\overset{\circ}{Z}) = 5$$

b) $\Lambda_{CD} = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$

This means that C and D are uncorrelated. Thus, using an affine estimator will not be useful at all. If the inputs are Gaussians and the outputs are Gaussians, the C and D are independent and output D gives you no information about output C.

$$4.19) \text{a)} E[(\bar{X}_t - \bar{X}_s)^2] \geq 0$$

If $E\bar{X}_t = E\bar{X}_0$ for all t

$$E(\bar{X}_t^2) = R_{\bar{X}}(t, t) = R_{\bar{X}}(0, 0) \text{ for all } t$$

$$E[(\bar{X}_t - \bar{X}_s)^2] \geq 0$$

$$E[\bar{X}_t^2 - 2\bar{X}_t \bar{X}_s + \bar{X}_s^2] \geq 0$$

$$E[\bar{X}_t^2] - 2E[\bar{X}_t \bar{X}_s] + E[\bar{X}_s^2] \geq 0$$

$$2R_{\bar{X}}(0, 0) \geq 2E[\bar{X}_t \bar{X}_s]$$

$$E[\bar{X}_t \bar{X}_s] \leq R_{\bar{X}}(0, 0)$$

$$R_{\bar{X}}(t, s) \leq R_{\bar{X}}(0, 0)$$

It is also true that

$$E[(\bar{X}_t + \bar{X}_s)^2] \geq 0$$

$$E[\bar{X}_t^2 + 2\bar{X}_t \bar{X}_s + \bar{X}_s^2] \geq 0$$

$$E[\bar{X}_t^2] + 2E[\bar{X}_t \bar{X}_s] + E[\bar{X}_s^2] \geq 0$$

$$2R_{\bar{X}}(0, 0) \geq -2E[\bar{X}_t \bar{X}_s]$$

$$-R_{\bar{X}}(t, s) \leq R_{\bar{X}}(0, 0)$$

If $R_{\bar{X}}(t, s) \leq R_{\bar{X}}(0, 0)$ and $-R_{\bar{X}}(t, s) \leq R_{\bar{X}}(0, 0)$

$$\text{then } |R_{\bar{X}}(t, s)| \leq R_{\bar{X}}(0, 0)$$

$$K_{\bar{X}}(t, s) = R_{\bar{X}}(t, s) - E[\bar{X}_t]E[\bar{X}_s] = R_{\bar{X}}(t, s) - (E\bar{X}_0)^2$$

$$K_{\bar{X}}(0, 0) = R_{\bar{X}}(0, 0) - E[\bar{X}_0]E[\bar{X}_0] = R_{\bar{X}}(0, 0) - (E\bar{X}_0)^2$$

$$R_{\bar{X}}(t, s) = K_{\bar{X}}(t, s) + (E\bar{X}_0)^2$$

$$R_{\bar{X}}(0, 0) = K_{\bar{X}}(0, 0) + (E\bar{X}_0)^2$$

$$R_{\bar{X}}(t, s) \leq R_{\bar{X}}(0, 0)$$

$$K_{\bar{X}}(t, s) + (E\bar{X}_0)^2 \leq K_{\bar{X}}(0, 0) + (E\bar{X}_0)^2$$

$$K_{\bar{X}}(t, s) \leq K_{\bar{X}}(0, 0)$$

$$K_{\bar{X}}(t, t) = R_{\bar{X}}(t, t) - E[\bar{X}_t]E[\bar{X}_t] = R_{\bar{X}}(0, 0) - E[\bar{X}_0]E[\bar{X}_0] = K_{\bar{X}}(0, 0)$$

$$E[(\bar{X}_t - E\bar{X}_t) - (\bar{X}_s - E\bar{X}_s)]^2 \geq 0$$

$$E[(\bar{X}_t - E\bar{X}_t)^2 - 2(\bar{X}_t - E\bar{X}_t)(\bar{X}_s - E\bar{X}_s) + (\bar{X}_s - E\bar{X}_s)^2] \geq 0$$

$$K_{\bar{X}}(t,t) - 2K_{\bar{X}}(t,s) + K_{\bar{X}}(s,s) \geq 0$$

$$2K_{\bar{X}}(0,0) - 2K_{\bar{X}}(+,s) \geq 0$$

$$K_{\bar{X}}(+,s) \leq K_{\bar{X}}(0,0)$$

$$E[(\bar{X}_t - E\bar{X}_t) + (\bar{X}_s - E\bar{X}_s)]^2 \geq 0$$

$$E[(\bar{X}_t - E\bar{X}_t)^2 + 2(\bar{X}_t - E\bar{X}_t)(\bar{X}_s - E\bar{X}_s) + (\bar{X}_s - E\bar{X}_s)^2] \geq 0$$

$$K_{\bar{X}}(+,+)+2K_{\bar{X}}(+,s)+K_{\bar{X}}(s,s) \geq 0$$

$$2K_{\bar{X}}(0,0) + 2K_{\bar{X}}(+,s) \geq 0$$

$$-K_{\bar{X}}(+,s) \leq K_{\bar{X}}(0,0)$$

If $K_{\bar{X}}(+,s) \leq K_{\bar{X}}(0,0)$ and $-K_{\bar{X}}(+,s) \leq K_{\bar{X}}(0,0)$ then

$$|K_{\bar{X}}(+,s)| \leq K_{\bar{X}}(0,0)$$

b) Autocorrelation and covariance functions are symmetric

$$R_{\bar{X}}(+,s) = E[\bar{X}_t \bar{X}_s] = E[\bar{X}_s \bar{X}_t] = R_{\bar{X}}(s,+)$$

$$K_{\bar{X}}(+,s) = E[\bar{X}_t \bar{X}_s] - E[\bar{X}_t]E[\bar{X}_s] =$$

$$E[\bar{X}_s \bar{X}_t] - E[\bar{X}_s]E[\bar{X}_t] = K_{\bar{X}}(s,+)$$