2 SCREW ROTATIONS

Purpose:

The purpose of this chapter is to introduce you to screw rotations. The screw rotation allows you to rotate a rigid body (or a frame representing the body pose) about an arbitrary axis in space and then determine the final pose of the body. This chapter also demonstrates that it is possible to move a body from any initial pose to any final pose with a single screw rotation and a proportional lead distance taken along a unique screw axis in space.

2.1 Rotation About an Arbitrary Axis

Previously we have rotated about either the x, y, or z axis of the base frame. Rotation about an arbitrary axis through the base origin with direction described by the \( \mathbf{k} \) unit vector (having components which are the direction cosines) can be determined by the transformation \( \mathbf{R}(\mathbf{k}, \theta) \) where

\[
\mathbf{R}(\mathbf{k}, \theta) = \begin{bmatrix}
k_x^2 v\theta + c\theta & k_y v\theta - k_z s\theta & k_z k_x v\theta + k_y s\theta \\
k_x k_y v\theta + k_z s\theta & k_y^2 v\theta + c\theta & k_z k_y v\theta - k_x s\theta \\
k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z^2 v\theta + c\theta
\end{bmatrix}
\]

and where

- \( k_x, k_y, \) and \( k_z \) = direction cosines of \( \mathbf{k} \)
- \( v\theta = 1 - c\theta \) \hspace{1cm} (versine of \( \theta \))
- \( s\theta = \sin \theta \)
- \( c\theta = \cos \theta \)

We will not drive (2.1), but show you the typical steps applied to derive (2.1). We arbitrarily select a frame \( x'y'z' \) such that its \( z' \) axis initially aligns with the \( \mathbf{k} \) unit vector direction. Thus, rotating about the \( \mathbf{k} \) axis is equivalent to rotating the \( x'y'z' \) axes described by \( \mathbf{C} \) relative to \( xyz \) (base frame) around the \( z' \) axis where \( \mathbf{C} \) is
\[
C = \begin{bmatrix}
    a_x & b_x & c_x & 0 \\
    a_y & b_y & c_y & 0 \\
    a_z & b_z & c_z & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\]

or

\[
C = \begin{bmatrix}
    a & b & c & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\]

and where

\[
a = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, \quad b = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}, \quad c = \begin{bmatrix} c_x \\ c_y \end{bmatrix} = k
\]

are the direction cosines of the \(x', y',\) and \(z'\) axes, respectively, with respect to the base coordinates.

Next, we rotate about the \(z'\) axis to the \(x''y''z''\) axes - see the following figure:

![Figure 2-2](image)

This operation can be described by the transformation \(H (R (z', \theta))\) where

\[
H = \begin{bmatrix}
    \cos \theta & -\sin \theta & 0 & 0 \\
    \sin \theta & \cos \theta & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\]

\(H\) describes the \(x''y''z''\) frame relative to the \(x'y'z'\) frame.

Now let \(u\) be a vector in \(xyz\) which has components \(v\) in \(x'y'z'\) before rotation about \(z'\) axis. Then

\[
u = Cv
\]
Now, rotating about the z’ axis, vector \( v \) rotates to vector \( w \). The components of \( v \) in x’y’z’ are the same as the components of \( w \) in x”y”z”. The components of \( w \) in the x’y’z’ frame are

\[
w = H v
\]  
(2.4)

To determine the coordinates of \( w \) in the xyz coordinates (call this vector \( r \)),

\[
r = C w = CH v
\]  
(2.5)

but

\[
v = C^{-1} u
\]  
(2.6)

so therefore

\[
r = CHC^{-1} u
\]  
(2.7)

Thus rotation about the z’ axis located by \( k = e \) rotates vector \( u \) to vector \( r \) by the transformation \( CHC^{-1} \). That (2.7) is equivalent to (2.1) can be shown by performing \( CHC^{-1} \) where

\[
C^{-1} = C^T = \begin{bmatrix}
a_x & a_y & a_z & 0 \\
b_x & b_y & b_z & 0 \\
c_x & c_y & c_z & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]  
(2.8)

and simplifying using the right hand relationship \( c = a \times b \) and \( k = e \) (see Paul1, pp 25-29).

### 2.2 Equivalent Angle and Axis of Rotation

Given (2.1) one might logically desire the \( k \) and \( \theta \) for the rotation. This inverse problem is not easy to solve and involves subtle trigonometric manipulations. After a rotation \( R(k,\theta) \), the xyz axes are transformed to a new set of axes described by

\[
R = [a \ b \ c]
\]  
(2.9)

where \( a, b, c \) are the direction cosines of the rotated axes with respect to the unrotated or base axes. Equating (2.9) to (2.1),

\[
[a \ b \ c] = R(k,\theta)
\]  
(2.10)

---

we determine 9 simultaneous equations for the solution of $k_x$, $k_y$, $k_z$ and $\theta$. This only involves 3 unknowns since $k_x^2 + k_y^2 + k_z^2 = 1$

Following the solution in Paul, pp 30-32, sum the diagonal terms

$$a_x + b_y + c_z = (k_x^2 + k_y^2 + k_z^2) v\theta + 3 c\theta = v\theta + 3 c\theta = 1 + 2 c\theta$$  \hspace{1cm} (2.11)

giving

$$\cos\theta = (a_x + b_y + c_z -1)/2$$  \hspace{1cm} (2.12)

Thus

$$\theta = \cos^{-1}((a_x + b_y + c_z -1)/2)$$  \hspace{1cm} (2.13)

Note that, at this point, equation (2.13) does not provide a unique $\theta$.

Next, we obtain expressions for $k_x$, $k_y$, $k_z$ by differencing pairs of off-diagonal terms in (2.10)

$$a_y - b_x = 2 k_z \sin \theta$$  \hspace{1cm} (2.14a)

$$c_x - a_z = 2 k_y \sin \theta$$  \hspace{1cm} (2.14b)

$$b_z - c_y = 2 k_x \sin \theta$$  \hspace{1cm} (2.14c)

Squaring and adding,

$$(a_y - b_x)^2 + (c_x - a_z)^2 + (b_z - c_y)^2 = 4\sin^2 \theta$$

giving,

$$\sin \theta = \pm \frac{1}{2} \sqrt{(a_y - b_x)^2 + (c_x - a_z)^2 + (b_z - c_y)^2}$$  \hspace{1cm} (2.15)

(2.15) again specifies a non-unique soln for $\theta$, but if we require that $k$ and $\theta$ be chosen such that the first solution is such that $0 \leq \theta \leq 180^o$, then $\sin \theta$ assumes the positive (+) value in (2.15). $k$ and $\theta$ can always be chosen such that $0 \leq \theta \leq 180^o$; see the geometric example below (simplified to rotation about the z axis).

$$\theta > 180^o$$  \hspace{1cm} Equivalent
With $0 \leq \theta \leq 180^\circ$, (2.12) and (2.15) provide a unique solution for $\theta$ since $\sin \theta$ and $\cos \theta$ identify the correct quadrant

$$\theta = \tan^{-1} \left( \frac{\sin \theta}{\cos \theta} \right)$$

(use atan2) (2.16)

Given the correct value, the components of $k$ may be determined using (2.14a) - (2.15c).

Numerical problems may exist as $\theta \to 0^\circ$ or $\theta \to 180^\circ$. For small $\theta \approx 0^\circ$, the numerator and denominators in

$$k_x = \frac{b_z - c_y}{2 \sin \theta}$$

(2.17a)

$$k_y = \frac{c_x - a_z}{2 \sin \theta}$$

(2.17b)

$$k_z = \frac{a_y - b_x}{2 \sin \theta}$$

(2.17c)

are both $\approx 0$ and thus ill defined. Paul suggests that $k$ be renormalized to ensure $|k| = 1$.

If $\theta \to 180^\circ$, then $\sin \theta \to 0$. Thus, we will apply a different soln for $\theta > 90^\circ$. First, we equate the diagonal elements in (2.10) to obtain,

$$k_x^2 \nu \theta + c \theta = a_x$$

(2.18a)
\[ k_y^2 \, v \theta + c \theta = b_y \]  \hspace{1cm} (2.18b)

\[ k_z^2 \, v \theta + c \theta = c_z \]  \hspace{1cm} (2.18c)

Expanding \( v \theta = 1 - \cos \theta \) and solving for \( k_x \), \( k_y \), \( k_z \),

\[
    k_x = \pm \sqrt{\frac{a_x - c \theta}{1 - c \theta}} 
\]  \hspace{1cm} (2.19a)

\[
    k_y = \pm \sqrt{\frac{b_y - c \theta}{1 - c \theta}} 
\]  \hspace{1cm} (2.19b)

\[
    k_z = \pm \sqrt{\frac{c_z - c \theta}{1 - c \theta}} 
\]  \hspace{1cm} (2.19c)

Since \( \sin \theta \geq 0 \), the proper radical signs for \( k_x \), \( k_y \), \( k_z \) can be determined from the sign of 
\( b_x - c_z \), \( c_x - a_z \), and \( a_y - b_x \), respectively, in equations (2.14a) - (2.14c).

For accuracy, only the largest \( k \) is determined from (2.19) -- Why?. The remaining \( k \) are determined by pairs of off-diagonal elements in (2.10) to get

\[
    a_y + b_x = 2 \, k_x k_y \, v \theta 
\]  \hspace{1cm} (2.20a)

\[
    b_z + c_y = 2 \, k_y k_z \, v \theta 
\]  \hspace{1cm} (2.20b)

\[
    a_z + c_x = 2 \, k_z k_x \, v \theta 
\]  \hspace{1cm} (2.20c)

These equations can be solved for the other \( k \), avoiding the square root calculations of (2.19) and avoiding numerical difficulties at \( \theta = 180^\circ \) \((v \theta = 2)\).

### 2.3 Other Transformations

**Euler Angles** \((\phi, \theta, \psi)\):

\[
    \begin{align*}
        \phi & \quad \theta & \quad \psi \\
        xyz & \rightarrow x'y'z' & \rightarrow x''y''z'' \\
    \end{align*}
\]

Euler angles describe any possible orientation by a sequence of 3 rotations, \( \phi \) about \( z \), \( \theta \) about \( y' \), and \( \psi \) about \( z'' \) as shown in Figure 2-4.
Now any vector in \( w \) in \( x''y''z'' \) axes can be described in base \( xyz \) axes after rotations \( \phi, \theta, \) and \( \psi \) by the following sequence of operations.

\[
\begin{align*}
\mathbf{v} &= \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{w} \\
\mathbf{u} &= \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \mathbf{v} \\
\mathbf{q} &= \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{u}
\end{align*}
\]

Thus, the coordinates \( \mathbf{q} \) of point \( w \) in base \( xyz \) axes after rotations \( \phi, \theta, \) and \( \psi \) are

\[
\mathbf{q} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{w}
\]

and performing the matrix multiplications,

\[
\mathbf{q} = \begin{bmatrix} c\phi c\theta c\psi -s\phi s\psi \\ s\phi c\theta c\psi + c\phi s\psi \\ -s\phi c\theta c\psi -s\phi s\psi \end{bmatrix} \begin{bmatrix} c\theta & c\phi & -s\phi s\psi \\ s\phi c\theta & c\phi & -s\phi s\psi \\ -s\phi c\theta & s\phi & c\theta \end{bmatrix} \mathbf{w}
\]

\tag{2.21}
The transformation sequence can be viewed relative to the base coordinates in the sequence $\psi$, $\theta$, and $\phi$ and written as

$$ q = R(\phi,z) \ R(\theta,y) \ R(\psi,x) $$

Roll, Pitch, Yaw ($\phi$, $\theta$, $\psi$):

The usual order is roll $\phi$ about $z$, pitch $\theta$ about $y$, and yaw $\psi$ about $x$

![Diagram of Roll, Pitch, Yaw rotations](image)

The transformation sequence to locate $w$ in base axes as vector $q$ is

$$ q = R(\phi,z) \ R(\theta,y) \ R(\psi,x) \ w $$

and can be multiplied to get

$$ q = \begin{bmatrix} c\phi \ c\theta & c\phi \ s\theta \ s\psi - s\phi \ c\psi & c\phi \ s\theta \ c\psi + s\phi \ s\psi \\ s\phi \ c\theta & s\phi \ s\theta \ s\psi + c\phi \ c\psi & s\phi \ s\theta \ c\psi - c\phi \ s\psi \\ -s\theta & c\theta \ s\psi & c\theta \ c\psi \end{bmatrix} \ w $$

(2.22)

**2.4 Screw Displacement**

$k$ as directed along $S$ defines the screw translation direction while $\theta$ defines the screw rotation. The screw translation along the $k$ direction, given the coordinates of $P$ and $P'$ (a point in the body being displaced to a different position), can be calculated by determining the plane $\perp$ to $k$ and containing point $P$. Define this plane by the equation

$$ n^T x = h $$

(2.23)

where $n = k$ if $h = k^T p > 0$ and $x$ is any point in the plane. If $h < 0$, then let $n = -k$ such that $h > 0$. Given this normal, "outward" form, the screw translation distance $d$ can be calculated from the projection distance of $P'$ onto the defined plane by

$$ d = |n^T p' - h| $$

(2.24)
(Note: $\mathbf{n}^T \mathbf{p}' - h$ may be negative due to the normal form of the plane.)

**Figure 2-6 Screw displacement of object**

*Special cases:*

If $k$ such that $d = 0$ then both $P$ and $P'$ lie in the plane and no translation is required, i.e., the screw displacement reduces to pure rotation only. If $\theta$ zero but $d \neq 0$, then the screw displacement reduces to pure translation only.

Locating the screw axis requires the frame locations of $\text{xyz}$ relative to $\text{XYZ}$ -- call this $\mathbf{C}$ -- and $x'y'z'$ relative to $\text{XYZ}$ -- call this $\mathbf{C}'$.

Given $\mathbf{C}$ and $\mathbf{C}'$ the frame locating $x'y'z'$ relative to $\text{xyz}$ is $\mathbf{C}^{-1} \mathbf{C}'$. The intersection point of the screw axis $S$ with the orthogonal plane can be determined by the procedures depicted in the following figure.
Let
\[ q = \text{intersection point of S with plane} \]
and
\[ v = \text{projection point of P'} \text{ onto plane described by } n^T x = h \text{ where } v = p' - d n \]
so that
\[ v = p' - (n^T p' - h) n \] (2.25)

Now given \( p \) and \( p' \), \( q \) can be located in global XYZ axes by referring to the following figure, a normal view of the plane of interest.

\[ \text{Figure 2-7 Determining a point on the screw axis} \]

\[ \text{Figure 2-8 Determining } q \text{ by the rotation triangle} \]
Define $L = |v - p| = \text{norm} = \sqrt{\sum \text{(coord differences)}^2}$

If $0^\circ < \theta < 180^\circ$ then $\theta$ can be located by determining the unit vector normal to the vector $v - p$ and lying in the plane. Call this unit vector $e_a$ where $a$ is the minimum distance between $q$ and the line between $p$ and $v$.

$$a = \frac{L}{2 \tan \frac{\theta}{2}} \quad (\theta \neq 0)$$

$e_a$ defined by

$$e_a = k \times e_L = k \times (v - p)/L \quad (2.26)$$

Given $e_a$, $q$ is determined by

$$q = ae_a + (p + v)/2 \quad (2.27)$$

For the special case $\theta = 180^\circ$, $q = (p + v)/2$.

For the special case $\theta = 0^\circ$, $q = p$.

### 2.5 Screw Transformation Summary

The screw transformation, a special form of the rotational sub-matrix $R$, represents the rotation about an arbitrary axis that passes through the origin of the reference frame.

A plane in space can be described by the simple equation $n^T x = d$ where $n$ is the plane normal, $x$ is any point in the plane, and $d$ is the minimum distance of the plane from the reference frame origin.

It is possible to move a body from any initial pose to any final pose with a single screw rotation and a proportional lead distance taken along a unique screw axis in space. This is referred to as the screw displacement.

Other transformations that are useful are Euler’s angles and roll-pitch-yaw. Euler’s angles are often used in the aerospace industries, whereas roll-pitch-yaw is used in the aircraft and shipping industries to describe motion of rigid bodies.