Screw Rotation and Other Rotational Forms

Purpose:

The primary purpose of this chapter is to introduce you to screw rotations. The screw rotation allows you to rotate a rigid body (or a frame representing the body pose) about an arbitrary axis in space and then determine the final pose of the body. This chapter also demonstrates that it is possible to move a body from any initial pose to any final pose with a single screw rotation and a proportional lead distance taken along a unique screw axis in space.
In particular, you will

1. Examine the form of the screw matrix (screw vector, screw angle).

2. See how it is derived.

3. Determine the inverse solution.

4. Extend the screw matrix to include displacement.

5. Prove that a general location change of a rigid body can be modeled as a single screw displacement (screw rotation about screw vector and translation along a screw axis).

6. Examine the Euler transformation set of three rotations.
Rotation About an Arbitrary Axis

Rotation about an arbitrary axis through the base origin with direction described by the k unit vector (having components which are the direction cosines) can be determined by the transformation:

\[ R(k, \theta) = \begin{bmatrix}
  k_x^2 \, v \, \theta + c \, \theta & k_y \, k_x \, v \, \theta - k_z \, s \, \theta & k_z \, k_x \, v \, \theta + k_y \, s \, \theta \\
  k_x \, k_y \, v \, \theta + k_z \, s \, \theta & k_y^2 \, v \, \theta + c \, \theta & k_z \, k_y \, v \, \theta - k_x \, s \, \theta \\
  k_x \, k_z \, v \, \theta - k_y \, s \, \theta & k_y \, k_z \, v \, \theta + k_x \, s \, \theta & k_z^2 \, v \, \theta + c \, \theta
\end{bmatrix} \]

where \( k_x, k_y, \) and \( k_z \) = direction cosines of k
\[ v \theta = 1 - c \theta \quad \text{(versine of } \theta) \]
\[ s \theta = \sin \theta \]
\[ c \theta = \cos \theta \]
Rotation About an Arbitrary Axis

\[ \begin{align*}
\text{X} & \quad \theta \\
\text{Y} & \quad \theta \\
\text{Z} & \quad \theta \\
\end{align*} \]
Screw matrix derivation

The derivation method is summarily given in Chapter 2 of the reading material, but it follows these steps:

1. Prime frame described by C
2. $H$ describes rotation about $z'$ to double prime frame

We can resolve a vector originally in frame C after rotation $H$ by the equation $CHC^{-1}$, which can be shown to reduce to the screw matrix.
Screw rotations...inverse problem

Given the screw matrix $R$, what is the screw angle(s) and what is the screw vector(s) $k$?

\[
R(k, \theta) = \begin{bmatrix}
  k_x^2 v \theta + c \theta & k_y k_x v \theta - k_z s \theta & k_z k_x v \theta + k_y s \theta \\
  k_x k_y v \theta + k_z s \theta & k_y^2 v \theta + c \theta & k_z k_y v \theta - k_x s \theta \\
  k_x k_z v \theta - k_y s \theta & k_y k_z v \theta + k_x s \theta & k_z^2 v \theta + c \theta 
\end{bmatrix}
\]

In other words, find the 4 unknowns $k_x, k_y, k_z,$ and $\theta$ given 9 known elements of $R$. 
Inverse solution

Express the known 3 x 3 matrix $R$ as $R = [a \ b \ c]$ where $a$, $b$, and $c$ are axes unit vectors expressed in their direction cosine components (e.g., $a_x$, $a_y$, $a_z$ are the components of $a$). There are 9 matrix components that are known.

First, sum the diagonal elements to eliminate the $k$ unknowns. This gives the equation for $\theta$:

$$\cos \theta = (a_x + b_y + c_z - 1)/2$$

Unfortunately, this does not provide a unique $\theta$. Why?
Inverse solution

Next, difference pairs of off-diagonal terms to get

\[ a_y - b_x = 2k_z \sin \theta \]
\[ c_x - a_z = 2k_y \sin \theta \]
\[ b_z - c_y = 2k_x \sin \theta \]

Now square and add, then solve for \( \sin \theta \) to get

\[ \sin \theta = \pm \frac{1}{2} \sqrt{(a_y - b_x)^2 + (c_x - a_z)^2 + (b_z - c_y)^2} \]

Use the atan2 function to get a unique \( \theta \). What is the atan2 function?

Why the \( \pm \)?
What does this mean?
Inverse solution

Now, given \( \cos \theta \) and \( \sin \theta \), we determine that there are two solutions for \( \theta \) and \( k \). Can you explain this? What does it mean graphically? Which solution would you normally select?
Inverse solution

Now, given $\theta$, we determine the solutions for $k$:

\[
\begin{align*}
 k_x &= \frac{b_z - c_y}{2 \sin \theta} \\
 k_y &= \frac{c_x - a_z}{2 \sin \theta} \\
 k_z &= \frac{a_y - b_x}{2 \sin \theta}
\end{align*}
\]

It is suggested that the equations be renormalized after applying. But there are problems if $\theta$ is near 0 or 180 degrees! If too close to 0 or 180, a different solution should be applied. See the notes for this alternative solution approach.
Example – Given the xyz frame is originally coincident with the base frame, what is the equivalent and minimum screw angle and the screw vector that will rotate it to the orientation described by R?

\[
R = \begin{bmatrix}
0.933 & 0.067 & 0.354 \\
0.067 & 0.933 & -0.354 \\
-0.354 & 0.354 & 0.866
\end{bmatrix}
\]

Soln:
\[
\theta = 30 \text{ deg} \\
k = [0.707 \ 0.707 \ 0]
\]
Screw displacement

Hypothesis: It is possible to move any rigid body from an initial pose to a second pose in space by a single screw rotation and an additional translation along a unique screw axis that is parallel to the screw vector.
Vector plane equation

A plane can be located in space by the vector equation

\[ n^T x = h \]

where \( n \) is an outward vector normal to the plane

\( x = \) any point in the plane

\( h = \) minimum distance of the plane origin of the frame of reference.
Vector plane equation cases

Examine the form of the equation

\[ d = n^T x - h \]

If \( x \) is in the plane, then \( d = 0 \).

If \( x \) is a point not in the plane, then \( d \neq 0 \), where \( d > 0 \) if \( x \) is on the positive side of the plane (on the side of the outward normal) and \( d < 0 \) if on the negative side of the plane.

Thus, given the plane equation, it is easy to determine the perpendicular distance of any point in space from the plane.
Screw displacement solution

Suppose \( C \) is a known homogeneous transformation that locates frame \( xyz \) (body frame) relative to \( XYZ \), while \( C' \) locates frame \( x'y'z' \) (body frame at new location) relative to \( XYZ \). Determine the screw axis \((q \text{ and } k)\), lead distance \((d)\), and screw rotation \((\theta)\) that will accomplish the screw displacement of the body.
Screw displacement solution

P and P' represent a point in the body being displaced to a different location. The screw translation (lead distance) can be calculated by projecting P' onto a plane \( \perp \) to \( k \) containing point P. The simplest approach is to place the origin of the xyz frame at P (thus, \( p = 0 \)), and also allow the plane \( \perp \) to \( k \) to contain P. By choosing XYZ to be the same as xyz, \( C \) becomes an identity matrix while \( C' \) poses the body in its final location relative to its initial location.
Screw displacement solution

$k$ and $\theta$ are determined using the rotational sub-matrix of $C'$ and the screw vector solution equations presented earlier! Now we need only determine $d$ and $q$. 
Screw displacement solution

The screw translation distance $d$ can be calculated from the projection distance of $P'$ onto the defined plane by

$$d = k^T p'$$

($d$ is one of required parameters)

where $h$ will be zero (why?).

The projection point is calculated from

$$v = p' - d k$$
Screw displacement solution

Now given $p = 0$ and $p'$, $q$ can be located in the XYZ (or xyz) axes by referring to the following figure, a normal view of the plane of interest.
Screw displacement solution

Now,

\[ L = |v| \]

If \(0^\circ < \theta < 180^\circ\) then \(\theta\) can be located by determining the unit vector normal to the vector \(v\) and lying in the plane. Call this unit vector \(e_a\) where \(a\) is the minimum distance between \(q\) and the vector \(v\). \(a\) is determined from

\[ a = \frac{L}{2} / \tan \left( \frac{\theta}{2} \right) \]
Screw displacement solution

e_a is defined by

\[ e_a = k \times e_L = k \times \frac{v}{L} \]

Given \( e_a \), \( q \) is determined by

\[ q = a \, e_a + \frac{v}{2} \quad (q \text{ is final parameter}) \]

This completes the simplified solution. A more extensive solution is presented in the course notes for the case where xyz is not aligned with XYZ.
Euler transformation

Euler angles describe any possible orientation by a sequence of 3 rotations, $\phi$ about $z$, $\theta$ about $y'$, and $\psi$ about $z''$ as shown in the figure.

Note: Euler angles are used in aerospace industry to describe spacecraft and satellite motion.

Roll, pitch, yaw transformations as described in the course notes are used in the aircraft and shipping industries.
Euler matrices

Now any vector in w in x''y''z'' axes can be described in base xyz axes after rotations φ, θ, and ψ by the following matrix operations.

\[
\begin{bmatrix}
\cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
w
\end{bmatrix}
\]

\[
\begin{bmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{bmatrix}
\begin{bmatrix}
v
\end{bmatrix}
\]

\[
\begin{bmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
u
\end{bmatrix}
\]
Euler matrices

Thus, the coordinates $q$ of point $w$ in base $xyz$ axes after rotations $\phi$, $\theta$, and $\psi$ are

$$q = \begin{bmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{bmatrix} \begin{bmatrix}
\cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{bmatrix} w$$

$$q = \begin{bmatrix}
c\phi \ c\theta \ c\psi & -s\phi \ s\psi & -c\phi \ c\theta \ s\psi & -s\phi \ c\theta \ s\psi \\
s\phi \ c\theta \ c\psi + c\phi \ s\psi & -s\phi \ c\theta \ s\psi + c\phi \ c\psi & s\phi \ c\theta \ s\psi + c\phi \ s\psi & s\phi \ c\theta \ s\psi
\end{bmatrix} w$$

We could also work the inverse problem here by determining an Euler angle set to orient a body given a known orientation matrix.
Screw and other transformations summary

The screw transformation, a special form of the rotational sub-matrix $R$, represents the rotation about an arbitrary axis that passes through the origin of the reference frame.

A plane in space can be described by the simple equation $n^T x = d$ where $n$ is the plane normal, $x$ is any point in the plane, and $d$ is the minimum distance of the plane from the reference frame origin.
Screw and other transformations

summary

It is possible to move a body from any initial pose to any final pose with a single screw rotation and a proportional lead distance taken along a unique screw axis in space. This is referred to as the screw displacement.

Other transformations that are useful are Euler’s angles and roll-pitch-yaw. Euler’s angles are often used in the aerospace industries, whereas roll-pitch-yaw is used in the aircraft and shipping industries to describe the motion of rigid bodies.