ME 510

Analogy between free surface and compressible flows

You should see by now that free surface flows (open channel flows) and compressible flows have many similarities. Probably the strongest similarity is that between the hydraulic jump in open channel flow and the normal shock wave in compressible flow. Both are strongly dissipative phenomena that occur over a relatively short distance. Both are extremely complicated in detail but can be satisfactorily analyzed by the control volume approach. Both convert the flow from one state (supercritical, supersonic) to another (subcritical, subsonic).

Another similarity is the phenomena of wave motions and their relevant dimensionless parameters. Weak free-surface gravity waves in open channel flow are similar to sound waves in compressible flow. The flow state is classified by the Froude number in open channel flow and Mach number in compressible flow. For Froude or Mach numbers less than unity, disturbances at a point are propagated to all parts of the flow, but for Froude or Mach numbers greater than unity, disturbances propagate downstream only.

A final illustration of the similarities between open channel flow and compressible flow is the comparison between the response of a frictionless open channel to a change of bottom elevation and the response of an isentropic gas flow to area change. Are all of these similarities simply coincidence or is there a formal analogy between the two types of flow? If there is a formal analogy, we can interpret measurements or analysis in one type of flow in terms of the other type. We can demonstrate the formal analogy by considering the equations of continuity, momentum, and energy. Consider liquid flow in a rectangular open channel and gas flow in a closed passage. The continuity equations are

Channel flow:

$$Q = Vyb = const. \tag{1}$$

Gas flow:

$$\dot{m} = \rho VA = const. \tag{2}$$

Or differentiating each we have

$$\frac{dV}{V} + \frac{dy}{y} + \frac{db}{b} = 0 \tag{3}$$

$$\frac{dV}{V} + \frac{d\rho}{\rho} + \frac{dA}{A} = 0 \tag{4}$$

For isentropic gas flow, the differential energy and momentum equations are identical. The energy momentum equation is

$$VdV + \frac{dp}{\rho} = 0 \tag{5}$$

Using

$$dp = \left.\frac{\partial p}{\partial \rho}\right)_s d\rho, \qquad a^2 = \left.\frac{\partial p}{\partial \rho}\right)_s, \qquad M^2 = \frac{V^2}{a^2} \tag{6}$$

we have for the energy/momentum equation for isentropic gas flow

$$\frac{d\rho}{\rho} = -M^2 \frac{dV}{V} \tag{7}$$

For open channel flow, the specific energy is

$$E = \frac{V^2}{2g} + y \tag{8}$$

For frictionless flow in a horizontal channel, E=constant and

$$\frac{VdV}{g} + dy = 0 \tag{9}$$

Using

$$C^2 = by \quad \text{and} \quad F_r^2 = V^2/C^2$$

Reduces the open channel energy equation to

$$\frac{dy}{y} = -F_r^{\,2} \frac{dV}{V} \tag{10}$$

Comparing Eqs. 1 and 2 and 3 Eqs. 3 and 4 suggest the following analogy between isentropic flow of a gas and frictionless flow in a horizontal rectangular channel.

- Mach number is analogous to Froude number
- Density is analogous to depth
- Area is analogous to channel width
- Mach waves/angles are analogous to Froude waves/angles
- Hydraulic jumps/bow waves are analogous to shock waves

Since the governing equations have identical form, the analogy is complete and formally correct. This analogy is put to use in a water table or water tunnel in which a free surface flow of water is used to simulate compressible flow of a gas. A water table can be used for flow visualization , with hydraulic jumps representing shock waves and depth representing density. Depth measurements in a water table can be used to deduce densities in the analogous compressible flow. A water table or tunnel is much cheaper to build than a supersonic wind tunnel and it is generally much easier to see what is occurring visually.