Linear Momentum Examples

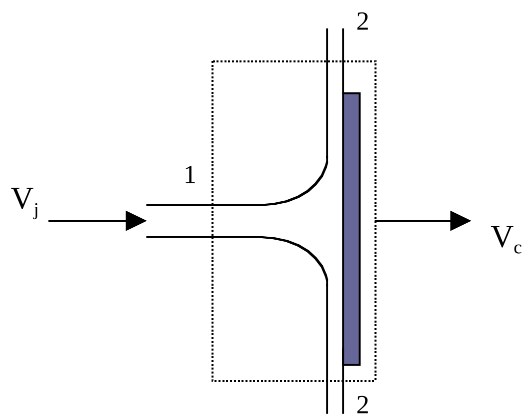
Examples utilizing the linear momentum principle to illustrate the use of:

1. Stationary control volumes
2. Control volumes translating at constant speed
3. Accelerating control volumes
4. Accelerating control volumes with changing mass.

Recall that the linear momentum principle in integral form is expressed as

$$\sum \vec{F} = \frac{\partial}{\partial t}\int_{c.v.} \rho \vec{V}_f \, d\forall + \int_{c.s.} \rho \vec{V}_s \left(\vec{V}_r \cdot \vec{n}\right) dA \tag{1}$$

where V_f represents the velocity of the fluid inside of the control volume with respect to an inertial coordinate system, V_s represents the velocity of the fluid at the control surfaces with respect to an intertial coordinate system, and V_r represents the velocity of the fluid at the control surfaces with respect to the control surface.



For the system shown below the velocity of the jet, V_j, is steady with respect to an inertial coordinate system. Determine the reaction force exerted on the control volume if

1. **V_c=0.**

The only force acting on the control volume is the reaction force and since there are no changes with time the unsteady term drops from Eq. (1). Writing the x component and assuming 1-D flow we get.

$$F_{Rx} = \int_{c.s.} \rho u \left(\vec{V} \cdot \vec{n}\right) dA = \rho V_j \left(-V_j\right) A_j + \rho(0)\left(V_j\right) A_j = -\rho V_j^2 A_j \tag{2}$$

2. **Now allow the plate to move at constant speed, V_c.**

Again there are no changes with time and the unsteady term drops. In a earth fixed reference frame the fluid velocity in x at 1 is V_j and at 2 the fluid velocity in x is the speed of the plate, V_c. Also with respect to the control surface at 1 the magnitude of the fluid velocity is V_j-V_c and at 2 it is this same amount. After substitution of these values Eq. 1 becomes

$$F_{Rx} = -\rho V_j \left(V_j - V_c\right) A_j + \rho V_c \left(V_j - V_c\right) A_j = -\rho \left(V_j - V_c\right)^2 A_j \qquad (3)$$

Note that this is the exact result that results if we transform the control volume to a stationary control volume and use relative velocities. Doing so the velocity in x at 1 is V_j-V_c and so also is the velocity term in the dot product. At 2 the velocity in x is now 0. The result will always be the same regardless if you transform and use the c.v. as the coordinate system or not since the c.v. is an inertial coordinate system.

3. Now allow the plate to accelerate and move at $V_c(t)$.

For an accelerating c.v. the unsteady term does not drop out. The velocities in x at 1 and 2 are exactly as above and the only difference is the addition of an unsteady term.

$$F_{Rx} = \frac{\partial}{\partial t}\left(m_{plate} V_c + \int_{c.v.} m_{fluid} V_{fluid} d\forall \right) - \rho \left(V_j - V_c\right)^2 A_j \qquad (4)$$

The unsteady term represents the acceleration of the plate and the fluid inside of the control volume. The plate is moving at speed V_c however the x component of velocity of the fluid inside of the c.v. varies with location, from V_j at 1 to V_c at 2. For most engineering applications to total mass of fluid in a c.v. is much much smaller than the mass of the moving object and is generally neglected. Thus the result is

$$F_{Rx} = m_{plate} \frac{dV_c}{dt} - \rho \left(V_j - V_c\right)^2 A_j \qquad (5)$$

If there is no reaction force the acceleration of the block is simply

$$\frac{dV_c}{dt} = \frac{\rho \left(V_j - V_c\right)^2 A_j}{m_{plate}} \qquad (6)$$

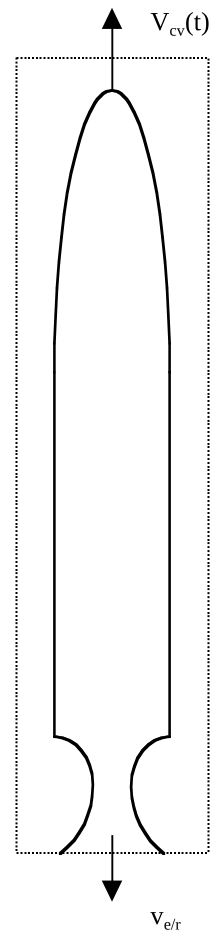
Note that in trying to solve the above problem incorrectly by using a coordinate system attached to the c.v. we would never end up with an acceleration term. Thus herein lies the importance of using the appropriate frame of reference. For stationary or steadily moving control volumes the mathematics will be simpler by using a reference frame on the c.v. However, if the c.v. is accelerating use a coordinate system fixed on earth.

4. Ascending Rocket Problem

Let us now consider the problem of an accelerating rocket with a changing mass (see the figure on the figure on the following page). The rocket is moving at $V_c(t)$ and the burned fuel exits the rocket at $V_{e/r}$ relative to the rocket.

Conservation of mass for the rocket says:

$$\frac{d}{dt}\left(M_{cv}\right)+\rho V_{e/r}A_e=0 \tag{7}$$



Since the rocket is accelerating, the velocity of the fluid at the exit with respect to earth is $V_c(t)$-$V_{e/r}$. Also the velocity of the fluid inside of the rocket is the same as the rocket, $V_c(t)$. The forces that act on the rocket are gravity, $M_{cv}g$, and aerodynamic drag, D. Making the above substitutions into (1) gives

$$-M_{cv}g-D=\frac{d}{dt}\left(M_{cv}V_c\right)+\rho\left(V_c-V_{e/r}\right)V_{e/r}A_e \tag{8}$$

Expanding we get

$$-M_{cv}g-D=M_{cv}\frac{d}{dt}V_c-\rho V_{e/r}^2A_e+V_c\left(\frac{d}{dt}M_{cv}+\rho V_{e/r}A_e\right) \tag{9}$$

Note however that the terms in brackets on the right hand side of Eq. (9) is the conservation of mass and is thus equal to zero. Making this simplification we get the rocket equation

$$-M_{cv}g-D=M_{cv}\frac{d}{dt}V_c-\rho V_{e/r}^2A_e \tag{10}$$

The $\rho V_{e/r}^2A_e$ term is generally referred to as the rockets thrust and can be increased by one of two means, either increasing the mass flux out of the nozzle or increasing the velocity of the gas, and preferably both. Eq. 10 is valid for low trajectory rockets, for rockets that travel high into or out of the atmosphere however, the earth can no longer be assumed an inertial coordinate system. In this case we can write a different equation that is with respect to the sun and it will include one additional acceleration term, the coriolis acceleration.