Class 16
Laplace Transforms

Reminder:
Dean’s Lecture tomorrow
11 am, JSB Auditorium
Dr. L. Douglas Smoot
Energy & Climate Change

Chapter 3
Laplace Transforms

• Important analytical method for solving linear ordinary
differential equations.
  - Application to nonlinear ODEs? Must linearize first.
• Laplace transforms play a key role in important process
control concepts and techniques.
  - Examples:
    • Transfer functions
    • Frequency response
    • Control system design
    • Stability analysis

[Image of mathematical equation]

Definition
The Laplace transform of a function, \( f(t) \), is defined as

\[
L[f(t)] = \int_0^\infty f(t)e^{-st} \, dt \quad (3-1)
\]

where \( F(s) \) is the symbol for the Laplace transform, \( L \) is the
Laplace transform operator, and \( f(t) \) is some function of time, \( t \).

Note: The \( L \) operator transforms a time domain function
\( f(t) \) into an \( s \)-domain function, \( F(s) \).

\( s \) is a complex variable: \( s = a + bj \), \( j = \sqrt{-1} \)

Inverse Laplace Transform, \( L^{-1} \):
By definition, the inverse Laplace transform operator, \( L^{-1} \),
converts an \( s \)-domain function back to the corresponding time
domain function:

\[
f(t) = L^{-1}[F(s)]
\]

Important Properties:
Both \( L \) and \( L^{-1} \) are linear operators. Thus,

\[
L[ax(t) + by(t)] = aL[x(t)] + bL[y(t)]
\]

where:
  - \( x(t) \) and \( y(t) \) are arbitrary functions
  - \( a \) and \( b \) are constants

\[
X(s) = L[x(t)] \quad \text{and} \quad Y(s) = L[y(t)]
\]

Similarly,

\[
L^{-1}[ax(s) + by(s)] = ax(t) + by(t)
\]

Laplace Transforms of Common
Functions
1. Constant Function
Let \( f(t) = a \) (a constant). Then from the definition of the
Laplace transform in (3-1),

\[
L[a] = \int_0^\infty ae^{-st} \, dt = \left. \frac{ae^{-st}}{-s} \right|_0^\infty = 0 - \left( \frac{a}{s} \right) = \frac{a}{s}
\]

(3-4)
2. Step Function

The unit step function is widely used in the analysis of process control problems. It is defined as:

\[ S(t) = \begin{cases} 
0 & \text{for } t < 0 \\
1 & \text{for } t \geq 0 
\end{cases} \quad (3-5) \]

Because the step function is a special case of a “constant”, it follows from (3-4) that

\[ L\left[ S(t) \right] = \frac{1}{s} \quad (3-6) \]

3. Derivatives

This is a very important transform because derivatives appear in the ODEs we wish to solve. In the text (p.41), it is shown that

\[ L\left[ \frac{df}{dt} \right] = s F(s) - f(0) \quad (3-9) \]

Initial condition at \( t = 0 \)

Similarly, for higher order derivatives:

\[ L\left[ \frac{d^n f}{dt^n} \right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0) \quad (3-14) \]

where:
- \( n \) is an arbitrary positive integer
- \( f^{(k)}(0) \) is the derivative of \( f \) evaluated at \( t = 0 \)

Special Case: All Initial Conditions are Zero

Suppose \( f(0) = f'(0) = \cdots = f^{(n-1)}(0) \). Then

\[ L\left[ \frac{d^n f}{dt^n} \right] = s^n F(s) \]

In process control problems, we usually assume zero initial conditions. Reason: This corresponds to the nominal steady state when “deviation variables” are used, as shown in Ch. 4.

4. Exponential Functions

Consider \( f(t) = e^{-bt} \) where \( b > 0 \). Then,

\[ L\left[ e^{-bt} \right] = \int_{0}^{\infty} e^{-bt} e^{-st} dt = \frac{1}{s + b} \quad (3-16) \]

5. Rectangular Pulse Function

It is defined by:

\[ f(t) = \begin{cases} 
0 & \text{for } t < 0 \\
1 & \text{for } 0 \leq t < t_w \\
0 & \text{for } t \geq t_w 
\end{cases} \quad (3-20) \]

The Laplace transform of the rectangular pulse is given by

\[ F(s) = \frac{1}{s} \left( 1 - e^{-t_w s} \right) \quad (3-22) \]

6. Impulse Function (or Dirac Delta Function)

The impulse function is obtained by taking the limit of the rectangular pulse as its width, \( t_w \), goes to zero but holding the area under the pulse constant at one. (i.e., \( \lim_{t_w \to 0} \frac{t_w}{t_w} = 1 \))

Let, \( \delta(t) \) be impulse function

Then, \( L\left[ \delta(t) \right] = 1 \)
Table 3.1. Laplace Transforms
See page 42 of the text.

<table>
<thead>
<tr>
<th>Laplace Transforms</th>
<th>(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Step function</td>
<td>1/s</td>
</tr>
<tr>
<td>2. Triangular wave</td>
<td>1/2(1 - 1/s^2)</td>
</tr>
<tr>
<td>3. Exponential decay</td>
<td>1/s^n</td>
</tr>
<tr>
<td>4. Sine wave</td>
<td>1/s^2</td>
</tr>
<tr>
<td>5. Cosine wave</td>
<td>s/s^2 + 1</td>
</tr>
<tr>
<td>6. Unit step function</td>
<td>1</td>
</tr>
<tr>
<td>7. Ramp function</td>
<td>1/s</td>
</tr>
<tr>
<td>8. Triangle wave</td>
<td>1/2(1 + 1/s)</td>
</tr>
<tr>
<td>9. Exponential growth</td>
<td>1/(s-1)</td>
</tr>
<tr>
<td>10. Cosine wave growth</td>
<td>1/(s^2 - 1)</td>
</tr>
</tbody>
</table>

Laplace table (cont.)

Practice
a. 1000 δ(t) (Step function with a magnitude of 1000)

\[ Y(s) = \frac{1000}{s} \]

b. 5e^{-6t} + sin 4t + 5

\[ Y(s) = 5\cdot\frac{1}{s+6} + \frac{4}{s^2+16} + \frac{5}{s} \]

c. \[ \frac{d^3y}{dt^3} \]

where
\[ \begin{align*}
\frac{dy}{dt}\bigg|_{t=0} &= 2, \\
\frac{d^2y}{dt^2}\bigg|_{t=0} &= 0, \\
y(0) &= 3
\end{align*} \]

Use #11 in Table 3.1

\[ Y(s) = 2 \left( \frac{5s^2 - 2s - 3}{s^3 - 3} \right) + \frac{5s^2 - 2s - 3}{s^3 - 3} \]

Partial Fraction Expansions

Solution of ODEs by Laplace Transforms
Procedure:
1. Take the L of both sides of the ODE.
2. Rearrange the resulting algebraic equation in the s domain to solve for the L of the output variable, e.g., Y(s).
3. Perform a partial fraction expansion.
4. Use the L^{-1} to find y(t) from the expression for Y(s).

Practice
Solve the following equation:
\[ \frac{dy}{dt} + 3y = e^{3t}, \quad y(0) = 2 \]

\[ sY(s) - y(0) = 2 + \frac{3}{s} \]

\[ Y(s) = 2 + \frac{3}{s} - \frac{1}{s+2} - \frac{1}{s+3} \]

Check Answer:
\[ \begin{align*}
y(0) &= T(t) = 2, \\
y(t) &= -2e^{2t} - 3e^{3t} \\
y(t) &= 3e^{2t} + 3e^{3t} \\
y(t) + 3y(t) &= e^{2t} 
\end{align*} \]

Partial Fraction Expansions

Basic idea: Expand a complex expression for Y(s) into simpler terms, each of which appears in the Laplace Transform table. Then you can take the L^{-1} of both sides of the equation to obtain y(t).

Example:
\[ Y(s) = \frac{s + 5}{(s+1)(s+4)} \]

Perform a partial fraction expansion (PFE)
\[ \frac{s + 5}{(s+1)(s+4)} = \frac{\alpha_1}{s+1} + \frac{\alpha_2}{s+4} \]

where coefficients \( \alpha_1 \) and \( \alpha_2 \) have to be determined.
To find $\alpha_1$: Multiply both sides by $s + 1$ and let $s = -1$

$$ \therefore \alpha_1 = \frac{x + 5}{s + 4} \bigg|_{s = -1} = \frac{4}{3} $$

To find $\alpha_2$: Multiply both sides by $s + 4$ and let $s = -4$

$$ \therefore \alpha_2 = \frac{x + 5}{s + 1} \bigg|_{s = -4} = -\frac{1}{3} $$
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More Practice

Practice: Write the Laplace form of a function that does the doublet test,
(a) changing at t = 0 to a value of 2,
(b) changing to a value of -2 at t = 3 min, and
(c) changing to a value of 0 at t = 6 min.

\[ \frac{2}{s} e^{-2\left(\frac{t}{s}\right)} - \frac{2}{s} e^{-6\left(\frac{t}{s}\right)} \]

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More Practice

Write the time domain form of the following Laplace function and sketch it:

\[ e^{-\frac{2}{s}} - \frac{2}{s} e^{-\frac{6}{s}} + \frac{3}{s} \]

\[ 3(e - 3[s(t - 3)] - 6[t - 6][s(t - 6)] + 3[t - 9][s(t - 9)] \]

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More Practice

Determine the final value of the following function:

\[ F(t) = \frac{12e^t + 22e + 6}{s(e + 1)[s + 2]} \]

\[ F(t = 0) = \frac{6}{(e + 1)[s + 2]} \]

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Extra

Example 3.1

Solve the ODE,

\[ \frac{dy}{dt} + 4y = 2 \quad y(0) = 1 \] (3.26)

First, take L of both sides of (3.26),

\[ s[Y(s) - 1] + 4Y(s) = \frac{2}{s} \]

Rearrange,

\[ Y(s) = \frac{\frac{2}{s} + s}{s + 4} \] (3.34)

Take L^{-1},

\[ y(t) = L^{-1}\left[ \frac{\frac{2}{s} + s}{s + 4} \right] \]

From Table 3.1,

\[ y(t) = 0.5e^{2t} + 0.5e^{\frac{t}{2}} \] (3.37)

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Partial Fraction Expansion

\[ \frac{y(t)}{L^{-1}} = \frac{\frac{5s + 2}{s(5s + 4)}}{x(s + 4)(x + 0.4)} \]

\[ \frac{5s + 2}{s(5s + 4)} = \frac{s + 0.4}{s(s + 0.8)} + \frac{s}{s(0.8)} + \frac{1}{s(0.8)} + \frac{0.4}{s(s + 0.8)} \]

\[ L^{-1}\left( \frac{1}{s(0.8)} + \frac{0.4}{s(s + 0.8)} \right) = e^{0.8t} + 0.4 \left[ e^{0.8t} - 1 \right] \]

\[ = e^{0.8t} - 0.4 \left[ e^{0.8t} - 1 \right] = 0.5 + 0.5e^{0.8t} \]
Example 3.2 (continued)

Recall that the ODE, \( y'' + 6y' + 11y = 0 \) with zero initial conditions resulted in the expression

\[
Y(s) = \frac{1}{s^3 + 6s^2 + 11s + 6} \quad (3-40)
\]

The denominator can be factored as

\[
s(s + 1)(s + 2)(s + 3) \quad (3-50)
\]

Note: Normally, numerical techniques are required in order to calculate the roots.

The PFE for (3-40) is

\[
Y(s) = \frac{a_1}{s + 1} + \frac{a_2}{s + 2} + \frac{a_3}{s + 3} \quad (3-51)
\]

Solve for coefficients to get

\[
\begin{align*}
a_1 &= \frac{1}{6}, \\
a_2 &= -\frac{1}{2}, \\
a_3 &= \frac{1}{2}, \\
a_4 &= -\frac{1}{6}
\end{align*}
\]

(For example, find \( a_1 \), by multiplying both sides by \( s \) and then setting \( s = 0 \).)

Substitute numerical values into (3-51):

\[
Y(s) = \frac{1/6}{s + 1} - \frac{1/2}{s + 2} + \frac{1/6}{s + 3}
\]

Take \( L^{-1} \) of both sides:

\[
L^{-1} [Y(s)] = \frac{1/6}{s} - \frac{1/2}{s+1} + \frac{1/2}{s+2} + \frac{1/6}{s+3}
\]

From Table 3.1,

\[
y(t) = \frac{1}{6} - \frac{1}{2} e^{-t} + \frac{1}{2} e^{-2t} - \frac{1}{6} e^{-3t} \quad (3-52)
\]

Important Properties of Laplace Transforms

1. Final Value Theorem

It can be used to find the steady-state value of a closed loop system (providing that a steady-state value exists).

Statement of FVT:

\[
\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s)
\]

providing that the limit exists (is finite) for all \( \text{Re} (s) \geq 0 \), where \( \text{Re} (s) \) denotes the real part of complex variable, \( s \).