Class 17

More Laplace

Complex Factors

- Denominator may have complex roots
  - $s^2 + d_1 s + d_0$ where $d_1^2/4 < d_0$
  - Remember quadratic formula $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Example: $s^2 + 4s + 5$ $\dfrac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1}$

$$(s+2+j)(s+2-j)$$
$$or \quad s = -2-j \;\; and \;\; -2+j$$

Implications of Complex Factors

- Complex roots indicate oscillatory behavior
- If the sign of the real part of the complex roots is negative, convergence is expected
  - Conversely, if the real part is positive, it will diverge
- Algebra needed to invert transforms with complex roots is messy but doable
- We don't need to invert the transform to tell whether it will converge or diverge, or whether or not it will oscillate

Practice

- Will y(t) converge or diverge? Is y(t) smooth or oscillatory?

$$Y(s) = \frac{s+2}{s(s^2+4s+13)}$$

Method 1: $s^2 + 4s + 13 = (s+2)^2 + 9 \Rightarrow oscillatory$

Method 2: $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{-4 \pm \sqrt{4^2 - 4 \cdot 13}}{2 \cdot 1} = \dfrac{-4 \pm \sqrt{-36}}{2} = -2 \pm 3j$

oscillatory, converging

Inverting Transforms with Complex Roots in the Denominator

- There are at least two different ways to proceed as described in your text:
  - Expansion without using complex numbers, followed by completing the square to invert the transform (preferred)
    - Example 3.4
  - Use of complex numbers and Euler's identity (p. 43)
    - $\cos(\omega t) = (e^{j\omega t} + e^{-j\omega t})/2$; $\sin(\omega t) = (e^{j\omega t} - e^{-j\omega t})/2$

Example 1

$$Y(s) = \frac{s+2}{s(s^2+4s+5)} = \frac{\alpha_1}{s} + \frac{\alpha_2 s + \alpha_3}{s^2+4s+5}$$

- Find α_1: $\alpha_1 = \left[\dfrac{s(s+2)}{s(s^2+4s+5)}\right]_{s=0} = \dfrac{2}{5}$
- To get α_2 and α_3, clear denominator and match "like" terms

$$s+2 = \alpha_1(s^2+4s+5) + s(\alpha_2 s + \alpha_3) = (\alpha_1+\alpha_2)s^2 + (4\alpha_1+\alpha_3)s + \alpha_1 5$$

- s^2 terms → $\alpha_1 + \alpha_2 = 0$, so $\alpha_2 = -2/5$
- s terms → $4\alpha_1 + \alpha_3 = 1$, so $\alpha_3 = -3/5$

$$Y(s) = \frac{2}{5s} + \frac{-\frac{2}{5}s - \frac{3}{5}}{s^2+4s+5}$$

Complete the square
Put into proper form for inversion

- Wanted: $s^2 + 4s + 5 = (s+b)^2 + w^2$
- How?
  b = (coefficient in front of s term)/2 = 4/2 = 2
- Knowing b, find w
  $b^2 + w^2 = 5 = 4 + w^2$, so w = 1

$$Y(s) = \frac{2}{5s} + \frac{-\frac{2}{5}s - \frac{3}{5}}{(s+2)^2 + 1}$$

Example 1 (cont)

Need to Get Form in Laplace Table

$$L\{e^{-bt}\cos(\omega t)\} = \frac{s+b}{(s+b)^2+\omega^2}$$

#15 in Table 3.1

$$L\{e^{-bt}\sin(\omega t)\} = \frac{\omega}{(s+b)^2+\omega^2}$$

#14 in Table 3.1

$$\frac{-\frac{2}{5}s - \frac{3}{5}}{(s+2)^2+1}$$

Has both an s and a number on the top

$$\frac{-\frac{2}{5}s - \frac{3}{5}}{(s+2)^2+1} = \frac{-\frac{2}{5}(s+2) + \frac{1}{5}}{(s+2)^2+1} = -\frac{2}{5}\left[\frac{(s+2)}{(s+2)^2+1}\right] + \frac{1}{5}\left[\frac{1}{(s+2)^2+1}\right]$$

Finally:

$$Y(s) = \frac{2}{5s} - \frac{2}{5}\left[\frac{(s+2)}{(s+2)^2+1}\right] + \frac{1}{5}\left[\frac{1}{(s+2)^2+1}\right]$$

and inverting $y(t) = \frac{2}{5} - \frac{2}{5}e^{-2t}\cos t + \frac{1}{5}e^{-2t}\sin t$

Example 1 (cont)

Analyze the Equation

$$y(t) = \frac{2}{5} - \frac{2}{5}e^{-2t}\cos t + \frac{1}{5}e^{-2t}\sin t$$

- e^{-t} terms mean that the system will converge at long time
- sin and cos terms mean permanent oscillations

Example 1 (cont)

One More Practice Problem

$$Y(s) = \frac{1}{s^2 - 4s + 13}$$

$$s^2 - 4s + 13 = (s-2)^2 + 9$$

$$Y(s) = \frac{1}{(s-2)^2 + 9} = \frac{1}{3}\frac{3}{(s-2)^2 + 3^2}$$

$$y(t) = \frac{1}{3}e^{2t}\sin(3t)$$

Oscillatory, diverges

What if Roots to Denominator Are:

$$\begin{bmatrix} 2+6i \\ 2-6i \\ -1 \\ -3 \\ -2 \end{bmatrix}$$

- $2+6i$: **Oscillatory, diverges**
- $2-6i$: **Oscillatory, diverges**
- -1: **No oscillations, converges**
- -3: **No oscillations, converges**
- -2: **No oscillations, converges**

Overall: Oscillatory, diverges

Initial Value

$$\frac{(s+2)}{(s+3)(s+4)}$$

Multiply by s and set s = ∞

$$\frac{s(s+2)}{(s+3)(s+4)} = \left[\frac{1\left(1+\frac{2}{s}\right)}{\left(1+\frac{3}{s}\right)\left(1+\frac{4}{s}\right)}\right]_{s\to\infty} = 1$$

Divide both top and bottom by s^2

Final Value

$$\frac{(s+6)}{(s+1)(s+2)}$$

Multiply by s and set s = 0

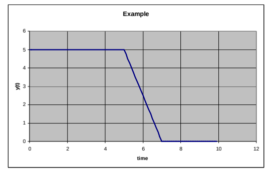
$$\frac{s(s+6)}{(s+1)(s+2)} = \left[\frac{s(s+6)}{(s+1)(s+2)}\right]_{s\to 0} = 0$$

Time Delay

(Fortran File)

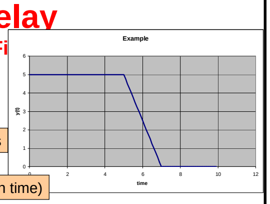
Wanted:

- Initial step to 5
- Ramp from 5 to 0 starting at t = 5 and ending at t = 7
- Final value of 0 after t = 7



Time Delay

(Fortran Fi




```
      program ft
      fun = 0.0
      S1=0.
      S2=0.
      S3=0.
      do 100 t=0.,10.,0.1
        if(t.ge.0.0)s1=1.
        if(t.ge.5.)s2=1.
        if(t.ge.7.)s3=1.
          fun=S1*5+(-5/2.)*(t-5.)*S2+5/2.*(t-7.)*S3
        print*,t,fun
 100  continue
      stop
      end
```

Initial values of step functions

Do Loop (step in time)

Step Functions