Complex Factors

- Denominator may have complex roots
  - $s^2 + d_1 s + d_0$ where $d_1^2/4 < d_0$
  - Remember quadratic formula
- Example: $s^2 + 4s + 5$

Implications of Complex Factors

- Complex roots indicate oscillatory behavior
- If the sign of the real part of the complex roots is negative, convergence is expected
  - Conversely, if the real part is positive, it will diverge
- Algebra needed to invert transforms with complex roots is messy but doable
- We don’t need to invert the transform to tell whether it will converge or diverge, or whether or not it will oscillate

Practice

- Will $y(t)$ converge or diverge? Is $y(t)$ smooth or oscillatory?

\[
Y(s) = \frac{s + 2}{s(s^2 + 4s + 13)}
\]

Method 1: $s^2 + 4s + 13 = (s + \_ \_)^2 + \_

Method 2: $-\frac{b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 13}}{2 \cdot 1}$
Inverting Transforms with Complex Roots in the Denominator

- There are at least two different ways to proceed as described in your text on pp. 48-49.
  - Use of complex numbers and Euler's identity
    \[ \cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}; \sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2} \]
  - Expansion without using complex numbers, followed by completing the square to invert the transform (preferred)
    - Example 3.4

**Example**

\[ Y(s) = \frac{s + 2}{s(5s^2 + 4s + 5)} = \frac{\alpha_1}{s} + \frac{\alpha_2 s + \alpha_3}{s^2 + 4s + 5} \]

- Find \( \alpha_1 \):
  - To get \( \alpha_2 \) and \( \alpha_3 \), clear denominator and match “like” terms
    - \( s^2 \) terms → \( \alpha_1 + \alpha_2 = 0 \), so \( \alpha_2 = -2/5 \)
    - \( s \) terms → \( 4 \alpha_1 + \alpha_3 = 1 \), so \( \alpha_3 = -3/5 \)

\[ Y(t) = \]

**Complete the square**

Put into proper form for inversion

- Wanted:
  \[ s^2 + 4s + 5 = (s + b)^2 + w^2 \]

- How?
  \[ b = (\text{coefficient in front of } s \text{ term})/2 = 4/2 = 2 \]

- Knowing \( b \), find \( w \)
  \[ b^2 + w^2 = 5 = 4 + w^2, \text{ so } w = 1 \]

**Need to Get Form in Laplace Table**

\[ \mathcal{L}[e^{-\alpha t}\cos(\omega t)] = \frac{s + b}{(s + b)^2 + \omega^2} \]
\[ \mathcal{L}[e^{-\alpha t}\sin(\omega t)] = \frac{\omega}{(s + b)^2 + \omega^2} \]

\[ \frac{s + 2}{(s + 2)^2 + 1} = \frac{-2(s + 2)}{(s + 2)^2 + 1} + 2 \int \frac{(s + 2)}{(s + 2)^2 + 1} \]
\[ = -2 \left[ \frac{1}{2} \right] \int \frac{(s + 2)}{(s + 2)^2 + 1} + \frac{1}{s + 2} \]

Finally:
\[ Y(s) = \frac{2}{5s} \left[ \frac{(s + 2)}{(s + 2)^2 + 1} \right] + \frac{1}{s + 2} \]

and inverting \( y(t) = \).
Analyze the Equation

\[ y(t) = \frac{2}{5} - \frac{2}{5} e^{-2t} \cos t + \frac{1}{5} e^{-2t} \sin t \]

- \( e^{-t} \) terms mean that the system will converge at long time
- \( \sin \) and \( \cos \) terms mean permanent oscillations

Whew!!

One More Practice Problem

\[ Y(s) = \frac{1}{s^2 - 4s + 13} \]

What if Roots to Denominator Are:

\[
\begin{bmatrix}
2 + 6i \\
2 - 6i \\
-1 \\
-3 \\
-2
\end{bmatrix}
\]

Initial Value

\[ \frac{(s + 2)}{(s + 3)(s + 4)} \]
**Final Value**

\[
\frac{(s + 6)}{(s + 1)(s + 2)}
\]

**Time Delay**

*(Fortran File)*

Wanted:
- Initial step to 5
- Ramp from 5 to 0 starting at \( t = 5 \) and ending at \( t = 7 \)
- Final value of 0 after \( t = 7 \)

```fortran
program ft
  fun = 0.0
  S1 = 0.
  S2 = 0.
  S3 = 0.
  do 100 t=0.,10.,0.1
    if(t.ge.0.0) S1=1.
    if(t.ge.5.) S2=1.
    if(t.ge.7.) S3=1.
    fun=S1*5+(-5/2.)*(t-5.)*S2+5/2.*(t-7.)*S3
  print*,t,fun
  100  continue
stop
end
```