Class 22
Complicated Transfer Functions

Business
• Next HW due Monday with SP10
  – Not Friday!!!
• Divide into lab groups (of 4)
  – Hand it to the TAs
• Sign up for tour of BYU Heating Plant
  – To see process control equipment up close
  – Class is too big!
  – This Friday (Oct. 22)
    • 1 pm
    • 2 pm
    • 3 pm

Road Map for 2nd Order Equations

Underdamped
$0 < \zeta < 1$
(5-51)
Critically damped
$\zeta = 1$
(5-50)
Overdamped
$\zeta > 1$
(5-48, 5-49)

Relationship between
$OS, P, t_1$ and $\zeta, \tau$
(pp. 119-120)

What About Higher Order Systems?

$G(s) = \frac{30}{24s^3 + 20s^2 + 10s + 2}$
or
$G(s) = \frac{30s^3 + 6s + 7}{24s^3 + 20s^2 + 10s + 2}$

Different Forms of $G(s)$

$G(s) = \frac{b_1(s-z_1)(s-z_2)...(s-z_n)}{a_1(s-p_1)(s-p_2)...(s-p_m)} e^{-\alpha s}$

so $z_1, z_2, ..., z_n$ are the zeros
and $p_1, p_2, ..., p_m$ are the poles

Alternatively, in time constant form,

$G(s) = \frac{\frac{b_1}{\tau_1}(s-z_1)\frac{b_2}{\tau_2}(s-z_2)...\frac{b_n}{\tau_n}(s-z_n)}{\frac{a_1}{\tau_1}(s-p_1)\frac{a_2}{\tau_2}(s-p_2)...\frac{a_m}{\tau_m}(s-p_m)} e^{-\alpha s}$

so $-1/\tau_1, -1/\tau_2, ..., -1/\tau_n$ are the zeros
and $-1/\tau_1, -1/\tau_2, ..., -1/\tau_n$ are the poles

Poles and Zeros
• Transfer function can usually be represented as a ratio of two polynomials in the Laplace variable $s$ along with a possible delay term:

$G(s) = \frac{Z(s)}{P(s)} e^{-\alpha s}$

where

$Z(s) = b_n s^n + b_{n-1} s^{n-1} + ... + b_1 s + b_0$

and

$P(s) = a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0$

Roots of $Z(s)$ = “zeros”
Roots of $P(s)$ = “poles”
So Who Cares?

- Poles show the stability of the process
- Zeros show some dynamics (lead-lag)
- Plot poles on real vs imaginary axes with "x"

What Do Zeros Tell Us?

- Zeros have no effect on system stability.
- Zero in right half plane: may result in an inverse response to a step change in the input

Example 6.2

For the case of a single zero in an overdamped second-order transfer function,

\[ G(s) = \frac{K(s + \frac{1}{\tau_1})}{(s + \frac{1}{\tau_2})(s + \frac{1}{\tau_3})} \]  

calculate the response to the step input of magnitude \( M \) and plot the results qualitatively.

Solution

The response of this system to a step change in input is

\[ y(t) = KM \left( 1 + \frac{\tau_2 - \tau_1}{\tau_1 - \tau_2} e^{-\frac{t}{\tau_2}} + \frac{\tau_2 - \tau_1}{\tau_1 - \tau_2} e^{-\frac{t}{\tau_1}} \right) \]

Chapter 6

Example Problem

\[ G(s) = \frac{30}{24s^4 + 20s^3 + 10s^2 + 2} = \frac{15}{12s^4 + 10s^3 + 5s + 1} = \frac{15}{(3s + 1)(4s^2 + 2s + 1)} \]

Put in pole-zero format:

\[ G(s) = \frac{15}{s + \frac{1}{3} \left( s^2 + \frac{1}{2} s + \frac{1}{4} \right)} \]

Convert to sine-cosine form:

\[ G(s) = \frac{15}{s + \frac{1}{3} \left( s^2 + \frac{1}{2} s + \frac{1}{4} \right)} = \frac{15}{s + \frac{1}{3} \left( s + \frac{1}{2} + \frac{1}{4} \right)} \]

Find poles:

\[ (-1/3, 0) \left\{ \begin{array}{c} 1 - \frac{\sqrt{3}}{4} \\ \frac{1}{4} - \frac{\sqrt{3}}{4} \end{array} \right\} \]
Chapter 6

Polynomial Approximations to $e^{-\theta s}$

Wanted: polynomial approximations to $e^{-\theta s}$

Why: Analysis of transfer functions

Two widely used approximations are:

1. Taylor Series Expansion:

$$e^{-\theta s} = 1 - \theta s + \frac{(\theta s)^2}{2!} - \frac{(\theta s)^3}{3!} + \frac{(\theta s)^4}{4!} + \ldots$$

(6-34)

The approximation is obtained by truncating after only a few terms.

2. Padé Approximations:

Many approximations are available. For example, the 1/1 approximation is,

$$e^{-\theta s} \approx \frac{1 - \frac{\theta s}{2}}{1 + \frac{\theta s}{2}}$$

(6-35)

Taylor Approximation of Higher-Order Transfer Functions

Goal: Approximate high-order transfer function models with lower-order models that have similar dynamic and steady-state characteristics.

- For small values of $s$,

$$e^{-\theta s} \approx 1 - \theta s$$

(6-57)

(use for numerator terms)

- An alternative first-order approximation consists of the transfer function,

$$e^{-\theta s} \approx \frac{1}{e^{\theta s}} = \frac{1}{1 + \theta s}$$

(6-58)

(use for denominator terms for non-dominant time constants)

Skogestad’s “half rule”

- Skogestad (2002) has proposed a related approximation method for higher-order models that contain multiple time constants.

- He approximates the largest neglected time constant in the following manner.

  - One half of its value is added to the existing time delay (if any) and the other half is added to the smallest retained time constant.

  - Time constants that are smaller than the “largest neglected time constant” are approximated as time delays using (6-58).
Skogestad Revisited
(sounds like a movie)

1. Find largest time constant (τ₁)
   • Keep it
2. Find 2nd largest time constant (τ₂)
   • Add half of τ₂ to τ₁
   • Add the other half of τ₂ to the time delay (in numerator)
3. All other τ’s
   • Add to time delay (in numerator)

Example please!

Example 6.4

Consider a transfer function:

\[ G(s) = \frac{K(-0.1s + 1)}{(5s + 1)(3s + 1)(0.5s + 1)} \]  \hspace{1cm} (6-59)

Derive an approximate first-order-plus-time-delay model,

\[ \tilde{G}(s) = \frac{Ke^{-0.5s}}{s + 1} \]  \hspace{1cm} (6-60)

using two methods:
(a) The Taylor series expansions of Eqs. 6-57 and 6-58.
(b) Skogestad’s half rule

Compare the normalized responses of \( G(s) \) and the approximate models for a unit step input.

Solution

(a) The dominant time constant (τ = 5) is retained. Applying the approximations in (6-57) and (6-58) gives:

\[ -0.1s + 1 \approx e^{-0.1s} \]  \hspace{1cm} (numerator) \hspace{1cm} (6-61)

and

\[ \frac{1}{3s + 1} \approx e^{-3s} \]  \hspace{1cm} (denominator term) \hspace{1cm} (6-62)

Substitution into (6-59) gives the Taylor series approximation, \( \tilde{G}_{TS}(s) \):

\[ \tilde{G}_{TS}(s) = \frac{Ke^{-0.1s} e^{-3s}}{5s + 1} = \frac{Ke^{-3.6s}}{5s + 1} \]  \hspace{1cm} (6-63)

and \( G(s) \) can be approximated as:

\[ \tilde{G}_{12}(s) = \frac{Ke^{-2.1s}}{6.5s + 1} \]  \hspace{1cm} (6-64)

(b) To use Skogestad’s method, we note that the largest neglected time constant in (6-59) has a value of three.

\[ G(s) = \frac{K(-0.1s + 1)}{(3s + 1)(0.5s + 1)} \]  \hspace{1cm} (6-59)

• According to his “half rule”, half of this value is added to the next largest time constant to generate a new time constant τ = 5 + 0.5(3) = 6.5.
• The other half provides a new time delay of 0.5(3) = 1.5.
• The approximation of the RHP zero in (6-61) provides an additional time delay of 0.1.
• Thus the total time delay in (6-60) is, \( 0 = 1.5 + 0.1 + 0.5 = 2.1 \).

Figure 6.10

Skogestad’s method provides better agreement with the actual response.
Parallel Process (cont.)

\[ \frac{Y(s)}{X(s)} = G_{\text{parallel}}(s) = G_1(s) + G_2(s) \]

\[ \frac{Y(s)}{X(s)} = \frac{K_1}{\tau_1 s + 1} + \frac{K_2}{\tau_2 s^2 + 2\zeta \tau_2 s + 1} \]

\[ = \frac{K_1(\tau_1^2 s^2 + 2\zeta \tau_1 s + 1) + K_2(\tau_2 s + 1)}{(\tau_1 s + 1)(\tau_2^2 s^2 + 2\zeta \tau_2 s + 1)} \]

\[ = \frac{K_1 \tau_2^2 s^2 + (K_1 + 2\zeta \tau_1) s + K_1 + K_2}{(\tau_1 s + 1)(\tau_2^2 s^2 + 2\zeta \tau_2 s + 1)} \]

Now put in standard form:

\[ = \frac{K' \left( ax^2 + bx + 1 \right)}{(\tau_1 s + 1)(\tau_2^2 s^2 + 2\zeta \tau_2 s + 1)} \]

Homework Hint on Prob 6.7

See online hint, because I changed the problem a little bit!

Moral:
2 systems in parallel give lead-lag and complicated pole-zero form