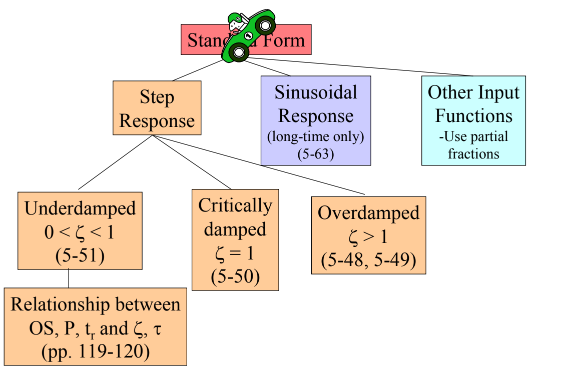
Class 22

Complicated Transfer Functions

Business

- Next HW due Monday with SP10
  - Not Friday!!!
- Divide into lab groups (of 4)
  - Hand it to the TAs
- Sign up for tour of BYU Heating Plant
  - To see process control equipment up close
  - Class is too big!
  - This Friday (Oct. 22)
    - 1 pm
    - 2 pm
    - 3 pm

Road Map for 2nd Order Equations



What About Higher Order Systems?

$$G(s) = \frac{30}{24s^3 + 20s^2 + 10s + 2}$$

or

$$G(s) = \frac{30s^2 + 6s + 7}{24s^3 + 20s^2 + 10s + 2}$$

A polynomial in the numerator is called a lead element

A polynomial in the denominator is called a lag element

WANTED!

Method to describe stability behavior of system using simple analysis of transfer function

Poles and Zeros

- Transfer function can usually be represented as a ratio of two polynomials in the Laplace variable *s* along with a possible delay term:

$$G(s) = \frac{Z(s)}{P(s)} e^{-\theta s}$$

where $Z(s) = b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0$

and $P(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$

Roots of Z(s) = "zeros"
Roots of P(s) = "poles"

Different Forms of G(s)

$$G(s) = \frac{b_m (s - z_1)(s - z_2)\ldots(s - z_m)}{a_n (s - p_1)(s - p_2)\ldots(s - p_n)} e^{-\theta s}$$

so $z_1, z_2, \ldots, z_n$ are the zeros
and $p_1, p_2, \ldots, p_n$ are the poles

Alternatively, in time constant form,

$$G(s) = \frac{b_m(-z_1)(-z_2)\ldots(-z_m)(\tau_{l1}s+1)(\tau_{l2}s+1)\ldots(\tau_{lm}s+1)}{a_n(-p_1)(-p_2)\ldots(-p_n)(\tau_1 s+1)(\tau_2 s+1)\ldots(\tau_n s+1)} e^{-\theta s}$$

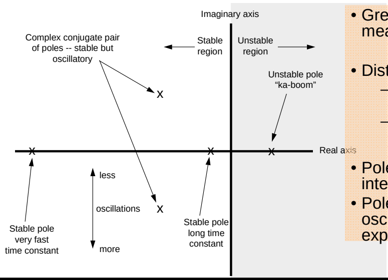
$$= K \frac{(\tau_{l1}s+1)(\tau_{l2}s+1)\ldots(\tau_{lm}s+1)}{(\tau_1 s+1)(\tau_2 s+1)\ldots(\tau_n s+1)} e^{-\theta s}$$

$n \geq m$ to be physically realizable

so $-1/\tau_{l1}, -1/\tau_{l2}, \ldots, -1/\tau_{ln}$ are the zeros
and $-1/\tau_1, -1/\tau_2, \ldots, -1/\tau_n$ are the poles

So Who Cares?

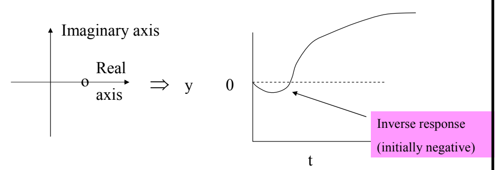
- Poles show the stability of the process
- Zeros show some dynamics (lead-lag)
- Plot poles on real vs imaginary axes with “×”



- Grey area (positive poles) means unstable behavior
- Distance from origin means
  - More oscillations (y direction)
  - Faster response due to shorter time constant (x direction)
- Pole on origin means integrating process
- Poles on y axis mean pure oscillatory behavior with no exponential damping

What Do Zeros Tell Us?

- Zeros have no effect on system stability.
- *Zero in right half plane:* may result in an inverse response to a step change in the input



- *Zero in left half plane:* may result in “overshoot” during a step response (see Fig. 6.3).

Chapter 6

Example 6.2

For the case of a single zero in an overdamped second-order transfer function,

$$G(s)=\frac{K(\tau_a s+1)}{(\tau_1 s+1)(\tau_2 s+1)} \qquad (6\text{-}14)$$

calculate the response to the step input of magnitude M and plot the results qualitatively.

Solution

The response of this system to a step change in input is

$$y(t)=KM\left(1+\frac{\tau_a-\tau_1}{\tau_1-\tau_2}e^{-t/\tau_1}+\frac{\tau_a-\tau_2}{\tau_2-\tau_1}e^{-t/\tau_2}\right) \qquad (6\text{-}15)$$

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Note that $y(t\to\infty)=KM$ as expected; hence, the effect of including the single zero does not change the final value nor does it change the number or location of the response modes. But the zero does affect how the response modes (exponential terms) are weighted in the solution, Eq. 6-15.

A certain amount of mathematical analysis (see Exercises 6.4, 6.5, and 6.6) will show that there are three types of responses involved here:

Case a: $\tau_a > \tau_1$

Case b: $0 < \tau_a \le \tau_1$

Case c: $\tau_a < 0$

Chapter 6

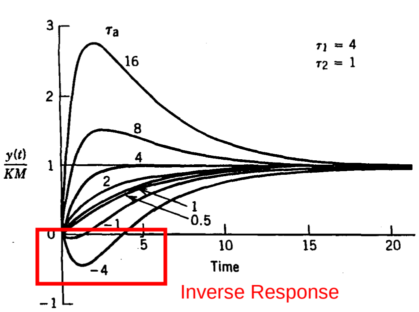


Figure 6.3. Step response of an overdamped second-order system (Eq. 6-14) with a single zero.

See page 134 for examples of inverse response
- Increase of steam to reboiler initially causes frothing/spillage on first trays

Example Problem

$$G(s)=\frac{30}{24s^3+20s^2+10s+2}\Rightarrow\frac{15}{12s^3+10s^2+5s+1}\Rightarrow\frac{15}{(3s+1)(4s^2+2s+1)}$$

Put in pole-zero format:

$$G(s)=\frac{15}{3\left(s+\frac{1}{3}\right)4\left(s^2+\frac{1}{2}s+\frac{1}{4}\right)}$$

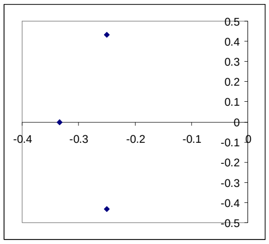
Convert to sine-cosine form:

$$G(s)=\frac{15}{3\left(s+\frac{1}{3}\right)4\left(s^2+\frac{1}{2}s+\frac{1}{4}\right)}=\frac{15}{12\left(s+\frac{1}{3}\right)\left[\left(s+\frac{1}{4}\right)^2+\frac{3}{16}\right]}$$

Find poles:

(-1/3,0) $\left(-\frac{1}{4},\frac{\sqrt{3}}{4}j\right)$ $\left(-\frac{1}{4},-\frac{\sqrt{3}}{4}j\right)$

Now Plot

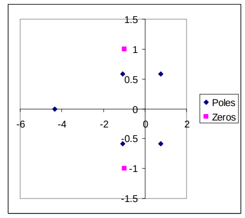


- All left-hand half
  - Exponential decay (good!)
- One imaginary conjugate pair
  - Oscillatory behavior

Example: Problem 6.1

$$G(s) = \frac{0.7\left(s^2 + 2s + 2\right)}{\left(s^5 + 5s^4 + 2s^3 - 4s^2 + 6\right)}$$

Find zeros and poles (use Mathcad)



Poles		
	-4.345	0.000
	0.756	0.583
	-1.083	0.585
	-1.083	-0.585
	0.756	-0.583
Zeros		
	-1	1
	-1	-1

- Two poles in unstable area (kaboom)
- Any input or disturbance action will cause growth beyond bounds

Polynomial Approximations to $e^{-\theta s}$

Wanted: polynomial approximations to

Why: Analysis of transfer functions

Two widely used approximations are:

1. Taylor Series Expansion: $e^{-\theta s}$:

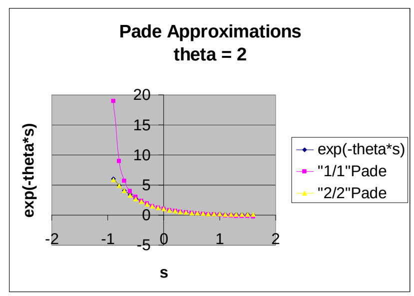
$$e^{-\theta s} = 1 - \theta s + \frac{\theta^2 s^2}{2!} - \frac{\theta^3 s^3}{3!} + \frac{\theta^4 s^4}{4!} + \dots \qquad (6\text{-}34)$$

The approximation is obtained by truncating after only a few terms.

2. Padé Approximations:

Many approximations are available. For example, the 1/1 approximation is,

$$e^{-\theta s} \approx \frac{1 - \frac{\theta}{2}s}{1 + \frac{\theta}{2}s} \qquad (6\text{-}35)$$



Implications for Control:

- Time delays are very bad for control because they involve a delay of information
- Pade approximation often used when $e^{-\theta s}$ is in denominator

Chapter 6

Taylor Approximation of Higher-Order Transfer Functions

Goal: Approximate high-order transfer function models with lower-order models that have similar dynamic and steady-state characteristics.

- For small values of s,

$$e^{-\theta_0 s} \approx 1 - \theta_0 s \qquad (6\text{-}57)$$

(use for numerator terms)

- An alternative first-order approximation consists of the transfer function,

$$e^{-\theta_0 s} = \frac{1}{e^{\theta_0 s}} \approx \frac{1}{1 + \theta_0 s} \qquad (6\text{-}58)$$

(use for denominator terms for non-dominant time constants)

Chapter 6

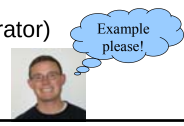
Skogestad's "half rule"

- Skogestad (2002) has proposed a related approximation method for higher-order models that contain multiple time constants.
- He approximates the largest neglected time constant in the following manner.
  - ➢One half of its value is added to the existing time delay (if any) and the other half is added to the smallest retained time constant.
  - ➢Time constants that are smaller than the "largest neglected time constant" are approximated as time delays using (6-58).

What does this mean?

Skogestad Revisited
(sounds like a movie)

1. Find largest time constant (τ_1)
   - Keep it
2. Find 2nd largest time constant (τ_2)
   - Add half of τ_2 to τ_1
   - Add the other half of τ_2 to the time delay (in numerator)
3. All other τ's
   - Add to time delay (in numerator)



Example 6.4

Consider a transfer function:

$$G(s)=\frac{K(-0.1s+1)}{(5s+1)(3s+1)(0.5s+1)} \qquad (6\text{-}59)$$

Derive an approximate first-order-plus-time-delay model,

$$\tilde{G}(s)=\frac{Ke^{-\theta s}}{\tau s+1} \qquad (6\text{-}60)$$

using two methods:

(a) The Taylor series expansions of Eqs. 6-57 and 6-58.

(b) Skogestad's half rule

Compare the normalized responses of $G(s)$ and the approximate models for a unit step input.

Solution

(a) The dominant time constant ($\tau = 5$) is retained. Applying the approximations in (6-57) and (6-58) gives:

$$-0.1s+1 \approx e^{-0.1s} \qquad (6\text{-}61)$$

(numerator)

and

$$\frac{1}{3s+1} \approx e^{-3s} \qquad \frac{1}{0.5s+1} \approx e^{-0.5s} \qquad (6\text{-}62)$$

(denominator terms)

Substitution into (6-59) gives the Taylor series approximation, $\tilde{G}_{TS}(s)$:

$$\tilde{G}_{TS}(s)=\frac{Ke^{-0.1s}e^{-3s}e^{-0.5s}}{5s+1}=\frac{Ke^{-3.6s}}{5s+1} \qquad (6\text{-}63)$$

(b) To use Skogestad's method, we note that the largest neglected time constant in (6-59) has a value of three.

$$G(s)=\frac{K(-0.1s+1)}{(5s+1)(3s+1)(0.5s+1)} \qquad (6\text{-}59)$$

- According to his "half rule", half of this value is added to the next largest time constant to generate a new time constant $\tau = 5 + 0.5(3) = 6.5$.
- The other half provides a new time delay of 0.5(3) = 1.5.
- The approximation of the RHP zero in (6-61) provides an additional time delay of 0.1.
- Approximating the smallest time constant of 0.5 in (6-59) by (6-58) produces an additional time delay of 0.5.
- Thus the total time delay in (6-60) is,

$$\theta = 1.5 + 0.1 + 0.5 = 2.1$$

and $G(s)$ can be approximated as:

$$\tilde{G}_{Sk}(s)=\frac{Ke^{-2.1s}}{6.5s+1} \qquad (6\text{-}64)$$

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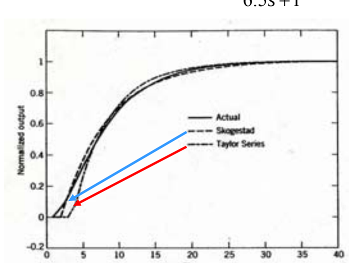
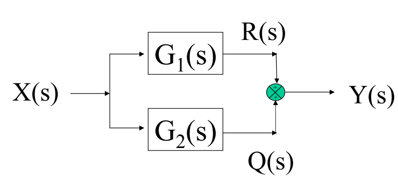


Figure 6.10 Comparison of the actual and approximate models for Example 6.4.

Skogestad's method provides better agreement with the actual response.

Example:
Parallel Processes



$G_1(s)$ is 1st order
$G_2(s)$ is 2nd order

Parallel Process (cont.)

$$\frac{Y(s)}{X(s)} = G_{overall}(s) = G_1(s) + G_2(s)$$

$$\frac{Y(s)}{X(s)} = \frac{K_1}{\tau_1 s + 1} + \frac{K_2}{\tau_2^2 s^2 + 2\zeta\tau_2 s + 1}$$

$$= \frac{K_1\left(\tau_2^2 s^2 + 2\zeta\tau_2 s + 1\right) + K_2\left(\tau_1 s + 1\right)}{\left(\tau_1 s + 1\right)\left(\tau_2^2 s^2 + 2\zeta\tau_2 s + 1\right)}$$

$$= \frac{K_1\tau_2^2 s^2 + \left(K_2 + 2\zeta\tau_2\right)s + K_1 + K_2}{\left(\tau_1 s + 1\right)\left(\tau_2^2 s^2 + 2\zeta\tau_2 s + 1\right)}$$

Now put in standard form:

$$= \frac{K'\left(as^2 + bs + 1\right)}{\left(\tau_1 s + 1\right)\left(\tau_2^2 s^2 + 2\zeta\tau_2 s + 1\right)}$$

Moral:
2 systems in parallel
give lead-lag and
complicated pole-zero form

Homework Hint on Prob 6.7

See online hint, because I
changed the problem a little bit!