Practice Exam 3 – Numerical Algebra

Ch En 263 – Numerical Tools

Instructions

- You have 50 minutes to complete the exam.
- You **may** use three pages (front and back) of notes
- You **may not** look at another person's exam or ask them for help, but you may of course ask clarifying questions to Dr. Tree or the TAs.
- You need a computer to complete this exam. You may not use a calculator.
- You may use scratch paper, but it will not be accepted for credit.
- Certain questions require that you submit an Excel workbook (`*.xlsx`) and/or a Python (`*.py`) file.
- **Save often!**
- **Make sure that you turn in the correct files!**

Exam Contents

This exam contains:

- 12 Qualitative Questions (36 pts)
- 4 Quantitative Questions (64 pts)
- Turn in your files (4 pts)

I. Qualitative Questions (36 pts)

Answer the indicated question with either True (T) or False (F), or the multiple choice letter as indicated.

1. _____(True or False) In root finding methods, one tries to find where the square of the residual is a minimum, but in optimization methods one tries to find where the residual crosses zero.
2. _____(True or False) In an iterative method, we converge to a solution when $|x^{(k+1)} - x^{(k)}| \to 0$ regardless of the value of the residual.
3. _____(True or False) The "cost" of a computation includes the memory that variables consume and calculations that consume the CPU time.
4. _____(True or False) When solving nonlinear "Engineering" equations, it is not necessary to put them in residual form because of the units (but it is still a good idea).
5. The system of linear equations

$$\begin{aligned} 5x - y + 3z &= 7 \\ x + 3y - 2z &= 1 \\ -2x - 2y + 5z &= -3 \end{aligned}$$

can be re-written in the form $\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{A}$ is a matrix, and $\boldsymbol{x}$ and $\boldsymbol{b}$ are vectors. Which are the correct $\boldsymbol{A}$ and $\boldsymbol{b}$ for the system shown above.

(a) $\boldsymbol{A} = \begin{bmatrix} 5 & 1 & -2 \\ -1 & 3 & -2 \\ 3 & -2 & 5 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} -7 \\ -1 \\ 3 \end{bmatrix}$

(b) $\boldsymbol{A} = \begin{bmatrix} 5 & 1 & -2 \\ -1 & 3 & -2 \\ 3 & -2 & 5 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 7 \\ 1 \\ -3 \end{bmatrix}$

(c) $\boldsymbol{A} = \begin{bmatrix} 5 & -1 & 3 \\ 1 & 3 & -2 \\ -2 & -2 & 5 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} -7 \\ -1 \\ 3 \end{bmatrix}$

(d) $\boldsymbol{A} = \begin{bmatrix} 5 & -1 & 3 \\ 1 & 3 & -2 \\ -2 & -2 & 5 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 7 \\ 1 \\ -3 \end{bmatrix}$

6. Which of the following is the correct formula for Newton's method for a single nonlinear equation:

(a) $x^{(k+1)} = x^{(k)} - f(x^{(k)})/f'(x^{(k)})$

(b) $x^{(k+1)} = x^{(k)} + f(x^{(k)})$

(c) $x_i = \dfrac{1}{a_{ii}} \left(b_i - \displaystyle\sum_{j=i+1}^{n-1} a_{ij}\, x_j \right)$

(d) $x_i^{(k+1)} = \dfrac{1}{a_{ii}} \left(b_i - \displaystyle\sum_{\substack{j=0 \\ j \neq i}}^{n-1} a_{ij}\, x_j^{(k)} \right)$

7. Which method would be most appropriate for solving this system of equations:

$$\begin{aligned} x &= 3y - \sin(7\pi/2) \\ 9x - y &= 12 \end{aligned}$$

(a) A fixed-point method

(b) Picard's method

(c) Newton's method

(d) Gauss Elimination

8. The Python function `outer_prod` takes two size-n arrays `a` and `b` as arguments and returns a matrix `c`. What is the asymptotic running time, $T(n)$ of this function for large n?

```
def outer_prod(a, b):
    c = np.zeros((n,n))
    for i in range(n):
        for j in range(n):
            c[i, j] = a[i]*b[j]
    return c
```

(a) $T(n) = \mathrm{O}(1)$ (b) $T(n) = \mathrm{O}(n)$ (c) $T(n) = \mathrm{O}(n^2)$ (d) $T(n) = \mathrm{O}(n^3)$

9. What are ways to come up with a good guess when solving a nonlinear equation? Select all that apply.

(a) A plot
(b) By taking the derivative
(c) Mathematical or physical bounds
(d) Physical intuition

10. A system of *nonlinear* equations where the number of equations is equal to the number of unknowns is guaranteed to have

(a) No solution
(b) One solution
(c) Multiple solutions
(d) There is no guarantee in general

11. Picards method is

(a) a simple, but sometimes unreliable method for solving a linear equation
(b) a simple, but sometimes unreliable method for solving a nonlinear equation
(c) a method for solving a linear equation that requires one to compute the derivative
(d) a method for solving a nonlinear equation that requires one to compute the derivative

12. Choose the answer that correctly writes the system of equations in standard/residual form

$$\begin{aligned} a^2 + 3b^2 &= 4 \\ a^2 + c^2 &= 1 \\ 2b^2 + c^2 &= 7 \end{aligned}$$

where $\boldsymbol{x} = [a, b, c]^T$.

(a) $f(x) = \begin{bmatrix} x_0^2 + 3x_1^2 - 4 \\ x_0^2 + x_2^2 - 1 \\ 2x_1^2 + x_2^2 - 7 \end{bmatrix}$ (b) $f(x) = \begin{bmatrix} x_0^2 + 3x_1^2 - 4 \\ x_0^2 + x_1^2 - 1 \\ 2x_0^2 + x_1^2 - 7 \end{bmatrix}$

(c) $f(x) = \begin{bmatrix} x_0^2 + 3x_1^2 \\ x_0^2 + x_2^2 \\ 2x_1^2 + x_2^2 \end{bmatrix}$ (d) $f(x) = \begin{bmatrix} x_0^2 + 3x_1^2 \\ x_0^2 + x_1^2 \\ 2x_0^2 + x_1^2 \end{bmatrix}$

II. Quantitative Questions (64 pts)

You must show your work for these problems in order to get full credit. For problems 13 and 14 use an Excel Workbook named "Lastname_Firstname_Exam3.xlsx". For problems 15 and 16 use a Python file named "Lastname_Firstname_Exam3.py".

13. Use Newton's method in Excel to find a value of t which satisfies this expression:

$$t^{1/2} - t = -\frac{1}{2}$$

14. The Van der Waals equation gives a relationship between the pressure, molar volume and temperature of a pure component fluid in either the gaseous or liquid state. A dimensionless version of this equation is given by

$$P_R = \frac{\frac{8}{3}T_R}{V_R - \frac{1}{3}} - \frac{3}{V_R^2}$$

where P_R is a dimensionless pressure, V_R is a dimensionless molar volume and T_R is a dimensionless temperature. Use the method of your choice in Excel to find at least one of the three molar volumes V_R that satisfy this equation when $T_R = 0.88$ and $P_R = 0.55$.

Hints: (i) There are no units in this problem. (ii) There is an asymptote at $V_R = 1/3$. The physically realistic values occur when $V_R > 1/3$.

15. Use Python to solve the system of equations. Report the value of z.

$$\begin{aligned} 5w + 7x - y + z &= -1 \\ 2w + 6x - 5y + 6z &= 18 \\ -w + 5x + 8y + 6z &= -8 \\ 2w - 5x - 8y - 4z &= 4 \end{aligned}$$

16. Use the method of your choosing to find a solution for s in Python.

$$s^5 + 3s^2 = 5$$