Lecture 20 — Integral Balances
Mass Balance

Class Business

- Reading: Deen 11.1–11.2
- Proposals due next time.

I. Macroscopic Analysis

Dimensional Analysis & phenomenology
- Fluid prop.
- Pipe flow
- Drag

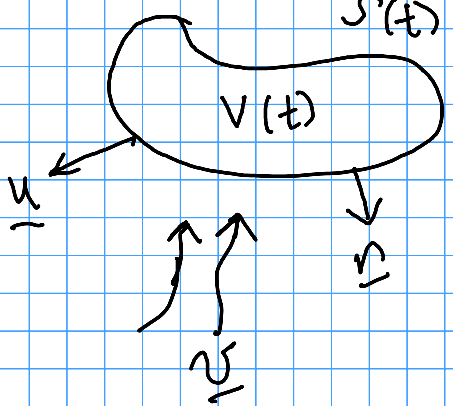
Differential Theory of fluid mech.
- statics
- kinematics
- Dynamics
  ↳ N-S!

★ What about Engineering?! Design?
- Empirics?
- Hard theory?
  - Turbulence
  - complex shapes
  - systems

★ Integral Balances help us solve these problems.

How? Burys messy details

II. Integral mass balance



V: volume — control vol.
S: surface " surface.
$\underline{n}$: normal vector
$\underline{u}$: velocity
$\underline{v}$: fluid velocity

How do we do a mass balance?

We already know <u>one</u> The continuity equation

$$\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho \underline{v}) = 0$$

p.295 Book has derivation

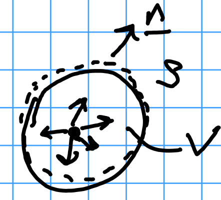
$$\int_{V(t)} \left[\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho \underline{v})\right] dV = 0$$

Get rid of these from inside of $\int_V dV$

- Gauss Div. Thm (GDT)
- Leibniz Integral rule (LIR)

GDT

$$\int_V \underline{\nabla}\cdot\underline{a}\, dV = \int_S \underline{n}\cdot\underline{a}\, dS$$



$$\int_V \underline{\nabla}\cdot(\rho \underline{v})\, dV = \int_S \underline{n}\cdot(\rho \underline{v})\, dS$$

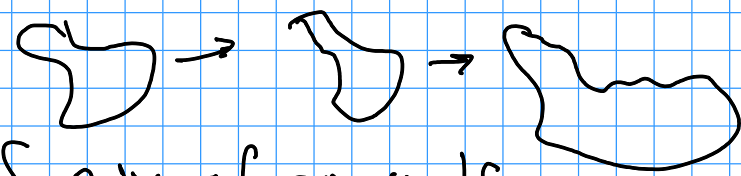
Eq. A.4-31

LIR

Eq. 11.2-17

$$\frac{d}{dt}\int_{V(t)} a\, dV = \int_{V(t)} \frac{\partial a}{\partial t}\, dV + \int_{S(t)} a\, \underline{n}\cdot\underline{u}\, dS$$

velocity of the surface.



$$\int_{V(t)} \frac{\partial \rho}{\partial t}\, dV = \frac{d}{dt}\int_{V(t)} \rho\, dV - \int_{S(t)} \rho\, \underline{n}\cdot\underline{u}\, dS$$

Final answer, combining all 3:

$$\frac{d}{dt}\int_{V(t)} \rho\, dV - \int_{S(t)} \rho\, \underline{n}\cdot\underline{u}\, dS + \int_{S(t)} \underbrace{\underline{n}\cdot(\rho\underline{v})}_{\rho\,\underline{n}\cdot\underline{v}}\, dS = 0$$

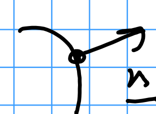
$$\frac{d}{dt}\underbrace{\int_{V(t)} \rho\, dV}_{m_{cv}} + \int_{S(t)} \rho\, \underline{n}\cdot\underbrace{(\underline{v}-\underline{u})}_{\underline{v}_{rel}}\, dS = 0$$

$$\boxed{\underbrace{\frac{dm_{cv}}{dt}}_{\text{accumulation term}} = \underbrace{-\int_{S(t)} \rho\, \underline{n}\cdot\underline{v}_{rel}\, dS}_{\text{in-out (net flux in)}}}$$

accumulation term

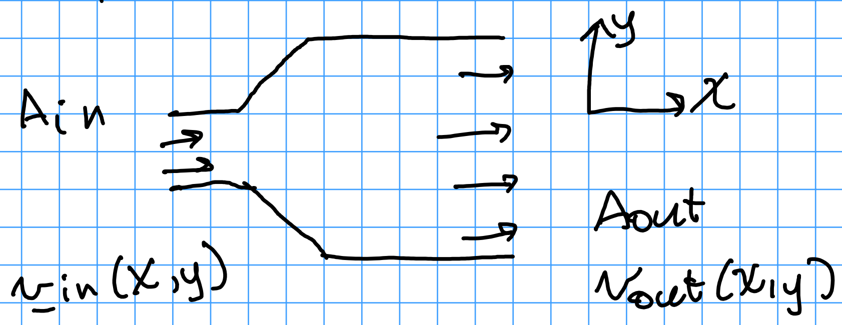
in-out (net flux in)

minus sign because $\underline{n}$ is out



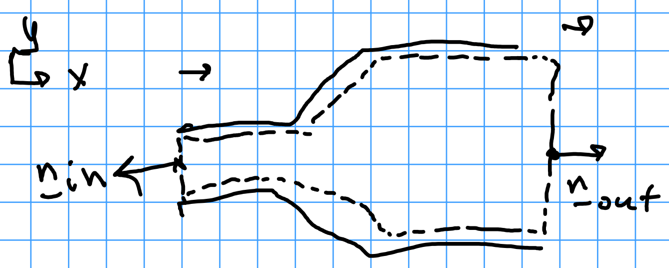
III. Examples

A. Diverging channel



* Given A_{in}, A_{out}, $v_{in}(x,y)$ can I find v_{out}?

① Draw the control volume



$$\frac{\partial}{\partial t}\int_{V(t)} \rho \, dV = -\int_{S(t)} \rho \, \underline{n} \cdot (\underline{v}-\underline{u}) \, dS$$

② Appropriately simplify

* steady $\longrightarrow \frac{\partial}{\partial t} \to 0$
* CV is fixed $\longrightarrow \underline{u} = 0$
* discrete inlet and outlet (simplify my integral)
* density is constant ($|\underline{v}| << c$)

$$0 = -\rho \int_{A_{in}} \underline{n}_{in} \cdot \underline{v}_{in} \, dA - \rho \int_{A_{out}} \underline{n}_{out} \cdot \underline{v}_{out} \, dA$$

$$\underline{n}_{in} \cdot \underline{v}_{in} = -v_{x,in}$$

$$\underline{n}_{out} \cdot \underline{v}_{out} = v_{x,out}$$

③ solve

$$0 = \int_{A_{in}} v_{x,in} \, dA - \int_{A_{out}} v_{x,out} \, dA$$

$$\langle v_{x,in} \rangle = \frac{1}{A_{in}} \int_{A_{in}} v_{x,in} \, dA$$

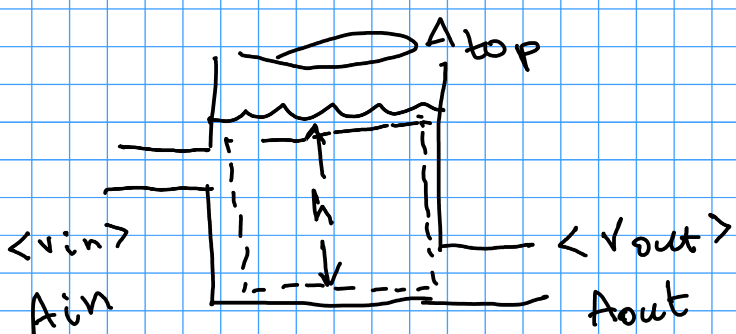
$$\langle v_{x,out} \rangle = \frac{1}{A_{out}} \int_{A_{out}} v_{x,out} \, dA$$

$$0 = \langle v_{x,in} \rangle A_{in} - \langle v_{x,out} \rangle A_{out}$$

$$\boxed{\langle v_{x,out} \rangle = \frac{A_{in}}{A_{out}} \langle v_{x,in} \rangle}$$

* we could only find $\langle v_{x,out} \rangle$, not $v_{out}(x,y)$

B. Tank Draining



What is $h(t)$?

$h(0) = h_0$

① Draw C.V.

② Make simplifications

* density is a constant
* discret inlet and outlet.

* CV is <u>not fixed</u> and <u>not steady</u>

$$\frac{d}{dt} \int_V \rho \, dV = -\int_{A_{in}} \rho (-v_{in}) \, dA - \int_{A_{out}} \rho (v_{out}) \, dA$$

$v_x = 0$

$$\frac{d}{dt}\underbrace{\int_V dV}_{V = A_{top} h} = + \langle v_{in} \rangle A_{in} - \langle v_{out} \rangle A_{out}$$

$$A_{top} \frac{dh}{dt} = \langle v_{in} \rangle A_{in} - \langle v_{out} \rangle A_{out}$$

ODE
$$\begin{cases} \dfrac{dh}{dt} = \dfrac{\langle v_{in} \rangle A_{in} - \langle v_{out} \rangle A_{out}}{A_{top}} \\ h(0) = h_0 \end{cases}$$

$$\boxed{h(t) = h_0 + \frac{\langle v_{in} \rangle A_{in} - \langle v_{out} \rangle A_{out}}{A_{top}} t}$$

The End.