Homework 5

Ch En 374 – Fluid Mechanics

Due date: 18 Oct. 2019

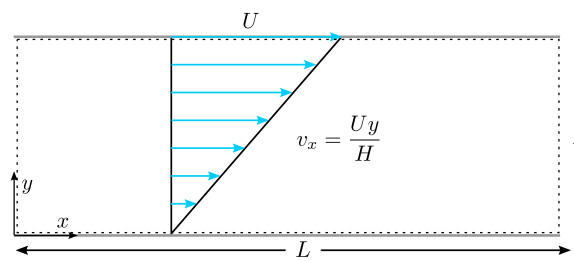
Survey Question

Please report how long it took you to complete this assignment (in hours) in the "Notes" section when you turn your assignment in on Learning Suite.

Practice Problems

1. [Lecture 14 – Kinematics I: Velocity Fields]. The figure below depicts flow in a horizontal parallel-plate channel in which $v_x = U$ at the upper wall and the lower wall is fixed. The wall spacing is H, the length under consideration is L, and the width (into the page) is W. This example of *simple shear flow* or *plane Couette flow* is something we have already seen several times. Assuming that the flow is steady, laminar and fully developed, the velocity is found to be,

$$v_x(y) = \frac{Uy}{H}$$
$$v_y = v_z = 0$$



   (a) Plot the velocity field in a Python notebook. Is the velocity field 1D, 2D, 3D and/or axisymmetric? *Hint: Use the Jupyter notebook I showed in class to help you with the Python plotting.*

   (b) Calculate the vorticity. Is this velocity field irrotational?

   (c) Generate an expression for the stream function. For convenience, let $\psi = 0$ along the bottom wall of the channel.

   (d) Calculate the volumetric flow rate, Q, in terms of U, H and W. How does the difference in the value of the stream function between the top wall and bottom wall, $\psi(x, H) - \psi(x, 0)$, compare to the volumetric flow rate, Q?

2. [Lecture 15 – Kinematics II: Acceleration and Continuity]. The v_x component of a steady, two-dimensional incompressible flow field is

$$v_x = 1 - \frac{y}{x + a}$$

   where a is an arbitrary constant. The velocity component v_y is unknown, but it is known that $v_y = 0$ at $y = 0$.

   (a) Generate an expression for v_y as a function of x and y.

   (b) Calculate the local acceleration and explain its significance in an Eulerian frame of reference.

   (c) Calculate the material derivative and explain its significance in Lagrangian frame of reference.

3. [Lecture 16 – Dynamics I: Stresses and Strain Rates]. Suppose that the force at point P in a fluid is determined by placing a small transducer there and measuring the force for different transducer orientations (the relevant area of the transducer is 0.001 m^2). The following time-independent results are shown on the right.

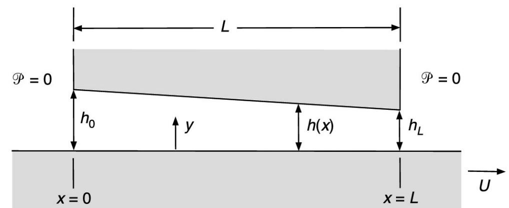
Surface orientation	Force (mN)
$\boldsymbol{e}_x$	$\boldsymbol{e}_x + 2\boldsymbol{e}_y$
$\boldsymbol{e}_y$	$2\boldsymbol{e}_x + 3\boldsymbol{e}_y - \boldsymbol{e}_z$
$\boldsymbol{e}_z$	$-\boldsymbol{e}_y + 4\boldsymbol{e}_z$

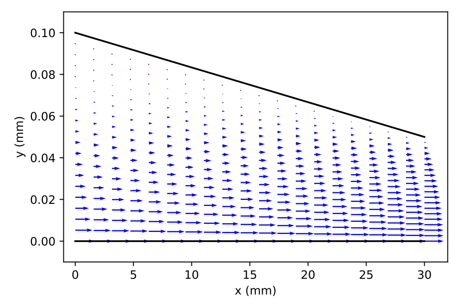
(a) Determine the Cartesian components of the total stress tensor, σ_{ij}.

(b) Find the force at point P for a surface with the orientation: $\boldsymbol{n} = \dfrac{1}{3}(\boldsymbol{e}_x + 2\boldsymbol{e}_y + 2\boldsymbol{e}_z)$.

Challenge Problems

4. [This problem is worth double]. Often, machine parts that are lubricated by oil are nearly parallel plates, as shown in the figure below. In this geometry, the bottom surface of length L and width W moves to the right with speed U and the top, slightly inclined surface at height h is stationary. Notice that the slight incline leads to a different velocity profile than the familiar parallel plate profile. The velocity and pressure fields are given by





$$v_x(x,y) = U\left[1 - 4\left(\frac{y}{h}\right)\left(1 - \frac{3\bar{h}}{2h}\right) + 3\left(\frac{y}{h}\right)^2\left(1 - \frac{2\bar{h}}{h}\right)\right]$$
$$v_y(x,y) = 2U\left(\frac{h_L - h_0}{L}\right)\left(\frac{3\bar{h}}{h} - 1\right)\left[\left(\frac{y}{h}\right)^2 - \left(\frac{y}{h}\right)^3\right]$$
$$P(x) = 6\mu UL\left[\frac{(h_0 - h)(h - h_L)}{(h_0^2 - h_L^2)h^2}\right]$$

where,

$$h = h(x) = \frac{h_L - h_0}{L}x + h_0 \quad \text{and} \quad \bar{h} = \frac{h_0 h_L}{h_0 + h_L}.$$

Note that gravitational forces are very small in lubrication applications and can therefore be ignored.

The goal of this problem is to get a force on the top surface when $U = 20$ mm/s, $h_0 = 0.1$ mm, $h_L = 0.05$ mm, $L = 30$ mm, $W = 1$ cm, and the oil has a viscosity of $\mu = 0.076$ Pa·s. To get this force, you will need the partial derivatives of the velocities evaluated at the top plate. You could derive expressions for these, but they involve messy algebra, so I did them for you :).

$$\left.\frac{\partial v_x}{\partial x}\right|_{y=h} = \frac{2U}{h}\frac{(h_L - h_0)}{L}\left(\frac{3\bar{h}}{h} - 1\right) \qquad \left.\frac{\partial v_x}{\partial y}\right|_{y=h} = \frac{2U}{h}\left(1 - \frac{3\bar{h}}{h}\right)$$
$$\left.\frac{\partial v_y}{\partial x}\right|_{y=h} = \frac{2U}{h}\left(\frac{h_L - h_0}{L}\right)^2\left(\frac{3\bar{h}}{h} - 1\right) \qquad \left.\frac{\partial v_y}{\partial y}\right|_{y=h} = \frac{2U}{h}\frac{(h_L - h_0)}{L}\left(1 - \frac{3\bar{h}}{h}\right)$$

Because the equations are long, this problem might appear intimidating. But, you can do this! The algebra is not as bad as it first appears. Report your answers numerically in SI units.

(a) Calculate the vorticity at $x = L/2$ and $y = h$.

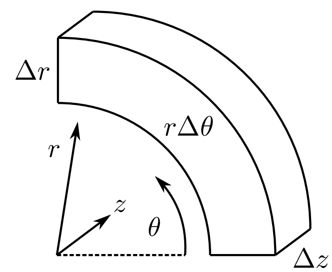
(b) Calculate the rate of strain tensor and use it to calculate the deformation (i.e. strain rate) of a material element oriented at 60° in the flow (i.e. $\boldsymbol{p} = [1/2, \sqrt{3}/2]^T$) at $x = L/2$ and $y = h$.

(c) Use Newton's law of viscosity to calculate the viscous stress tensor in Cartesian coordinates at $x = L/2$ and $y = h$. Which components are normal stresses and which are shear stresses?

(d) Calculate the total stress tensor in Cartesian coordinates at $x = L/2$ and $y = h$.

(e) Calculate the x and y components of the force/area (i.e. the local stress) on the top of the plate at $x = L/2$ and $y = h$.

5. In this problem, we are going to use a shell mass balance on the portion of a cylinder shown to the right to derive the continuity equation in cylindrical coordinates. Start by write down the general balance equation:



$$\text{accumulation} = \text{in} - \text{out} + \text{generation}$$

(a) (Accumulation) Write down the accumulation term in terms of the volume of the shape on the right. Find the volume of the shape in terms of r, Δr, $\Delta\theta$ and Δz. *Hint: The area of a semicircle is* $\frac{\theta}{2}r^2$. *Also, because* Δr *is very small* $(\Delta r)^2 \approx 0$, *and terms that have* $(\Delta r)^2$ *in them can be neglected.*

(b) (In - out) Calculate the mass flux in and out of the six faces of the cylindrical chunk by finding the mass flux in or out of the face times the area of the face. Be careful of negative signs and the directions of the unit vectors. *Hint: Some of the faces are rectangular, others are semicircular. You will again need to set terms with* (Δr^2) *in them to zero.*

(c) Combine all of the terms into your balance equation. What happens to the generation term? Divide by the volume and take the limit as Δr, $\Delta\theta$ and Δz go to zero to find the differential equation. *Hint: The final equation is in table* 5.1 *in your book*