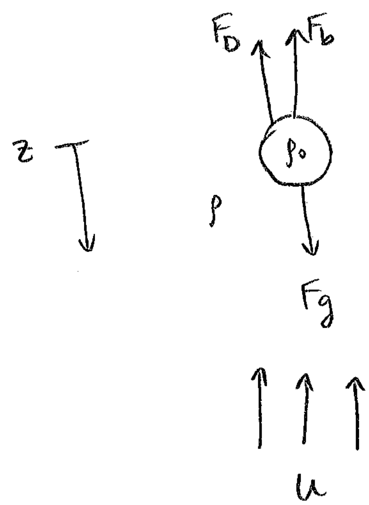
Question About Intuition on time to reach terminal velocity

Sept. 22, 2017

Q: Is it "volume" or "mass" that we should think makes a particle "bigger", and therefore slower to reach terminal velocity

A:

A transient force balance: (at high Re)



$$F_D = \frac{1}{2}\rho u^2 C_D A_\perp$$

$$F_b = \rho g V$$

$$F_g = \rho_0 g V$$

$$m^* \frac{du}{dt} = \sum_i F_i = F_g - F_D - F_b$$ (sum of forces in z-dir)

$$u(0) = 0$$

note about m^*: This is an effective mass due to the fluid the sphere carries with it initially.

$$m^* = \rho_0 V + \frac{1}{2}\rho V$$

* Substitute our expressions,

$$(\rho_0 V + \tfrac{1}{2}\rho V)\frac{du}{dt} = \rho_0 g V - \tfrac{1}{2}\rho u^2 c_D A_\perp - \rho g V \quad , \quad u(0)=0$$

* divide by ρV,

$$\left(\frac{\rho_0}{\rho} + \tfrac{1}{2}\right)\frac{du}{dt} = \frac{\rho_0}{\rho} g - \frac{1}{2}\frac{u^2 c_D A_\perp}{V} - g$$

$$\left(\frac{\rho_0}{\rho} + \tfrac{1}{2}\right)\frac{du}{dt} = \left(\frac{\rho_0}{\rho} - 1\right) g - \frac{c_D}{2}\frac{A_\perp}{V} u^2$$

← note: at high Re, c_D is a constant, so we are justified in leaving it alone here.

* Assume there are 2 scales: $\mathcal{U}$ and t_0

let $\theta = u/\mathcal{U}$

and $\tau = t/t_0$

← These are our definition. We are free to make them what we want

$$\left(\frac{\rho_0}{\rho} + \frac{1}{2}\right)\frac{\mathcal{U}}{t_0}\frac{d\theta}{d\tau} = \left(\frac{\rho_0}{\rho} - 1\right) g - \frac{c_D}{2}\frac{A_\perp}{V}\mathcal{U}^2\theta^2 \quad , \quad \theta(0) = 0$$

* divide by $(\frac{\rho_0}{\rho} - 1) g$

$$\underbrace{\frac{(\frac{\rho_0}{\rho} + \frac{1}{2})\mathcal{U}}{(\frac{\rho_0}{\rho} - 1) g t_0}}_{A}\frac{d\theta}{d\tau} = 1 - \underbrace{\frac{c_D A_\perp \mathcal{U}^2}{2V(\frac{\rho_0}{\rho} - 1) g}}_{B}\theta^2 \quad , \quad \theta(0)=0$$

Lets define τ and t_0 so $A=1$ and $B=1$.

* B first:

$$\frac{C_D A_\perp \tau^2}{2V(\frac{\rho_0}{\rho}-1)g} = 1 \quad \Rightarrow \quad \tau^2 = \frac{2V(\frac{\rho_0}{\rho}-1)g}{C_D A_\perp}$$

* Now A

$$\frac{(\frac{\rho_0}{\rho}+\frac{1}{2})\tau}{(\frac{\rho_0}{\rho}-1)g\,t_0} = 1 \quad \Rightarrow \quad t_0 = \frac{(\rho_0/\rho+1/2)}{(\rho_0/\rho-1)g}\,\tau$$

$$= \frac{(\rho_0/\rho+1/2)}{(\rho_0/\rho-1)g}\left[\frac{2V(\rho_0/\rho-1)g}{C_D A_\perp}\right]^{1/2}$$

$$= (\rho_0/\rho+1/2)\left[\frac{2V}{(\rho_0/\rho-1)g\,C_D A_\perp}\right]^{1/2}$$

* Back to our ODE for a minute:

$$\frac{d\theta}{d\tau} = 1-\theta^2 \qquad \theta(0)=0$$

* Solving ...

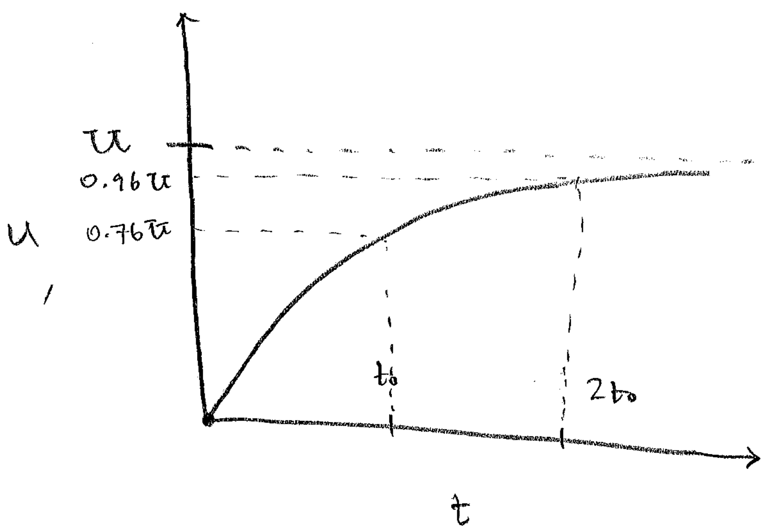
$$\frac{d\theta}{(1-\theta^2)} = d\tau \qquad \theta(0)=0$$

$$\tanh^{-1}(\theta) = \tau + C \qquad \tanh^{-1}(0) = 0$$

so, $C = 0$

$$\Rightarrow \theta = \tanh(\tau)$$

$$\Rightarrow \boxed{u = u_t \tanh(t/t_0)}$$



So, when $t = 2t_0$, u is 96% of the way to being u_t. This means u_t is the terminal velocity. t_0 should give us our intuition. Let's look at it again.

$$t_0 = \left[\frac{(\rho_0/\rho + 1/2)^2 \, 2V}{(\rho_0/\rho - 1) g \, c_D A_\perp} \right]^{1/2}$$

This is a little complicated. we often care about the case where:

$\rho_0 >> \rho$ (the object is much more dense than the liquid – think a ball falling in air)

$\Rightarrow \frac{\rho_0}{\rho} >> 1$

$\Rightarrow (\frac{\rho_0}{\rho} - 1) \approx \rho_0/\rho$

$\therefore (\frac{\rho_0}{\rho} + 1/2) \approx \rho_0/\rho$

So:

$$t_0 \approx \left[\frac{(\rho_0/\rho)^2 \, 2V}{(\frac{\rho_0}{\rho}) g \, c_D A_\perp} \right]^{1/2}$$

this is the object mass

$t_0 \uparrow$, $m_0 \uparrow$

$$\approx \left[\frac{2 \rho_0 V}{\rho g \, c_D A_\perp} \right]^{1/2} \approx \left[\frac{2}{c_D} \frac{\rho_0}{\rho} \frac{V}{A_\perp g} \right]^{1/2}$$

this is proportional to F_{Drag}. $t_0 \downarrow$ $F_D \uparrow$

$$\frac{V}{A_\perp} = \frac{D_H}{4} \leftarrow \text{hydraulic radius}$$

$$\boxed{t_0 \approx \left[\frac{1}{2c_D} \frac{\rho_0}{\rho} \frac{D_H}{g} \right]^{1/2}}$$

* t_0 goes up as D_H and the object density goes up and goes down for shapes with a larger drag coefficient.

$\Rightarrow$ More mass makes it take longer to reach terminal velocity, more drag makes it shorter.