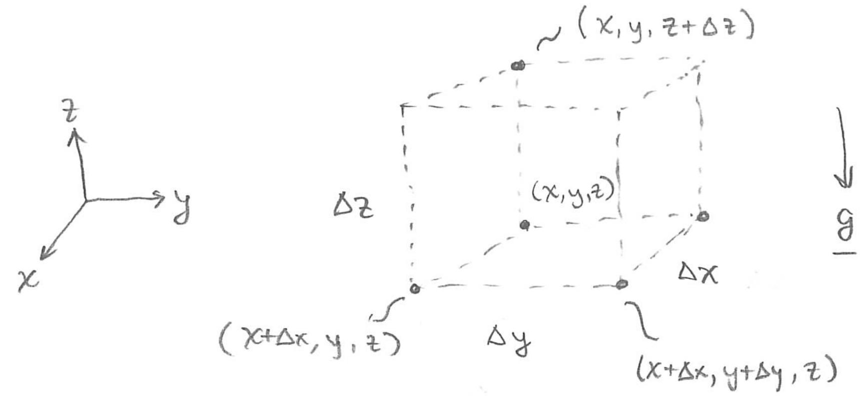
Derivation of the static Pressure Equation

* parallels Deen's derivation on pp. 85-87 in §4.2.

* Consider a rectangular prism of static fluid:



* We are going to do a force balance on this rectangular prism.

- Two types of terms:
  - Force from pressure on each face:

$$\underline{F} = -\underline{n} PA$$

6 faces: $\pm \underline{e}_x$, $\pm \underline{e}_y$, $\pm \underline{e}_z$

  - weight of the fluid:

$$\underline{F} = m\underline{g}$$

$$\sum_i \underline{F}_i = 0 \quad \text{(static fluid)}$$

$$-(-\underline{e}_x) P(x,y,z)\,\Delta y \Delta z - \underline{e}_x P(x+\Delta x, y, z)\,\Delta y \Delta z$$

$$-(-\underline{e}_y) P(x,y,z)\,\Delta x \Delta z - \underline{e}_y\, P(x, y+\Delta y, z)\,\Delta x \Delta z$$

$$-(-\underline{e}_z) P(x,y,z)\,\Delta x \Delta y - \underline{e}_z P(x,y,z+\Delta z)\,\Delta x \Delta y$$

$$-\underline{e}_z \rho g\,\Delta x \Delta y \Delta z = 0$$

(Phew!)

* divide by the volume of the prism: $\Delta x \Delta y \Delta z$

$$\frac{P(x,y,z) - P(x+\Delta x, y, z)}{\Delta x} \underline{e}_x$$

$$+ \frac{P(x,y,z) - P(x, y+\Delta y, z)}{\Delta y} \underline{e}_y$$

$$+ \frac{P(x,y,z) - P(x,y,z+\Delta z)}{\Delta z} \underline{e}_z - \rho g \underline{e}_z = 0$$

* Take the limit $\Delta x \to 0$, $\Delta y \to 0$ $\Delta z \to 0$

$$\lim_{\Delta z \to 0} \frac{P(x,y,z+\Delta z) - P(x,y,z)}{\Delta z} = \frac{\partial P}{\partial z}$$

$$\underbrace{-\frac{\partial P}{\partial x} \underline{e}_x - \frac{\partial P}{\partial y} \underline{e}_y - \frac{\partial P}{\partial z} \underline{e}_z}_{\underline{\nabla} = \frac{\partial}{\partial x} \underline{e}_x + \frac{\partial}{\partial y} \underline{e}_y + \frac{\partial}{\partial z} \underline{e}_z} \underbrace{- \rho g \underline{e}_z}_{\underline{g} = -g \underline{e}_z} = 0$$

$$\boxed{\underline{\nabla} P = \rho \underline{g}}$$

$$\boxed{\frac{\partial P}{\partial x} = 0 \qquad \frac{\partial P}{\partial y} = 0 \qquad \frac{\partial P}{\partial z} = -\rho g}$$

Static pressure equation

* we just did a differential force balance. This is sometimes called a "shell" balance because the shape we use is usually a thin shell.