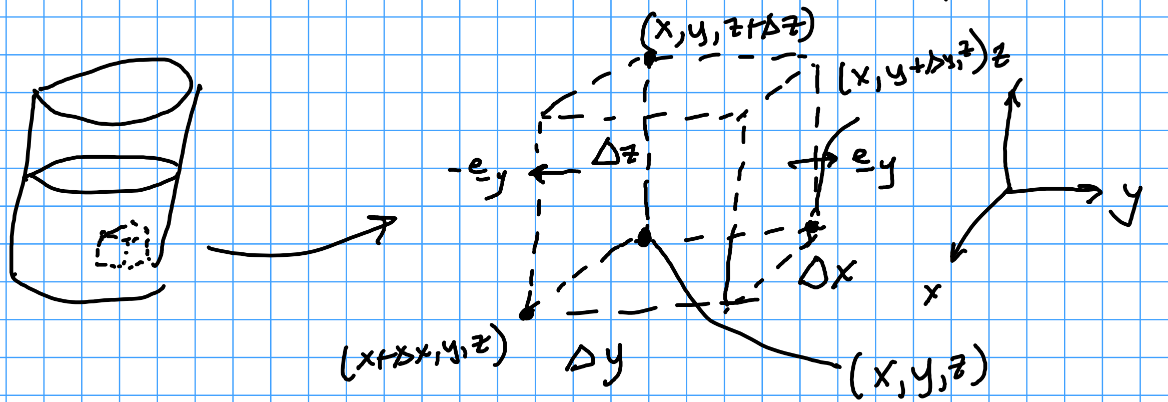
Lecture 12: Fluid Statics – Supplemental

Derivation of the fluid static equation

* Parallel the derivation in Deen §4.2, pp 85–87



$$\underline{g} = -g\,\underline{e}_z = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}$$

* Force Balance on the rectangular prism
  - Force from pressure on each side (6 sides)

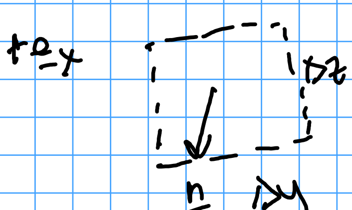
surface forces $\longrightarrow$ $\underline{F} = -\underline{n}PA$ $\qquad \underline{n} = \pm\underline{e}_x, \pm\underline{e}_y, \pm\underline{e}_z$

$P(x,y,z) \qquad A = \Delta y\Delta z$ or $\Delta x \Delta y$

body force $\longrightarrow$ – Weight of the fluid

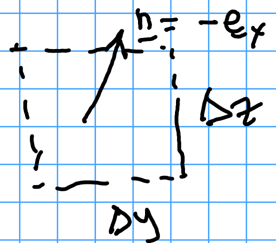
$$\underline{F} = m\underline{g}$$

$$\sum_i \underline{F}_i = 0 \quad \leftarrow \text{static fluid}$$



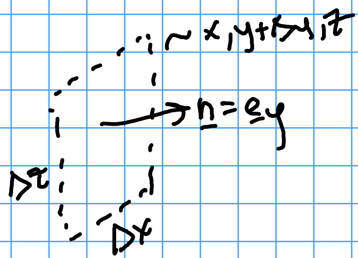
$$\underline{F} = -\underline{n}PA = -(\underline{e}_x)\,P(x+\Delta x, y, z)\,\Delta y\,\Delta z$$

$-\underline{e}_x$



$$\underline{F} = -\underline{n}\,PA = -(-\underline{e}_x)\,P(x,y,z)\,\Delta y\,\Delta z$$

$\underline{e}_y$



$$\underline{F} = -\underline{n}\,PA = -(\underline{e}_y)\,P(x, y+\Delta y, z)\,\Delta x\,\Delta z$$

$-\underline{e}_y$

$$\underline{F} = -\underline{n}\,PA = -(-\underline{e}_y)\,P(x,y,z)\,\Delta x\,\Delta z$$

$\underline{e}_z$

$$\underline{F} = -\underline{n}\,PA = -(\underline{e}_z)\,P(x,y,z+\Delta z)\,\Delta x\,\Delta y$$

$-\underline{e}_z$

$$\underline{F} = -\underline{n}\,PA = -(-\underline{e}_z)\,P(x,y,z)\,\Delta x\,\Delta y$$

Weight

$$\underline{F} = m\underline{g} = -mg\,\underline{e}_z = -\rho V g\,\underline{e}_z$$
$$= -\rho\,\Delta x\,\Delta y\,\Delta z\,g\,\underline{e}_z$$

(phew!)

$$-\underline{e}_x P(x+\Delta x, y, z)\,\Delta y\,\Delta z + \underline{e}_x P(x,y,z)\,\Delta y\,\Delta z \leftarrow 1/\Delta x$$
$$-\underline{e}_y P(x, y+\Delta y, z)\,\Delta x\,\Delta z + \underline{e}_y P(x,y,z)\,\Delta x\,\Delta z \leftarrow 1/\Delta y$$
$$-\underline{e}_z P(x, y, z+\Delta z)\,\Delta x\,\Delta y + \underline{e}_z P(x,y,z)\,\Delta x\,\Delta y \leftarrow 1/\Delta z$$
$$-\rho g\,\Delta x\,\Delta y\,\Delta z\,\underline{e}_z = 0$$

*Divide by volume: $\Delta x\,\Delta y\,\Delta z$

$$-\underline{e}_x\left[P(x+\Delta x,y,z) - P(x,y,z)\right]/\Delta x$$
$$-\underline{e}_y\left[P(x,y+\Delta y,z) - P(x,y,z)\right]/\Delta y$$

$$- \underline{e}_z \left[P(x,y,z+\Delta z) - P(x,y,z) \right] / \Delta z$$

$$- \rho g \underline{e}_z = 0$$

* Take the limit as $\Delta x \to 0$, $\Delta y \to 0$, $\Delta z \to 0$

$$\lim_{\Delta x \to 0} \frac{P(x+\Delta x, y, z) - P(x,y,z)}{\Delta x} = \frac{\partial P}{\partial x}$$

$$-\frac{\partial P}{\partial x} \underline{e}_x - \frac{\partial P}{\partial y} \underline{e}_y - \frac{\partial P}{\partial z} \underline{e}_z - \rho g \underline{e}_z = \underline{0}$$

$$\boxed{+\frac{\partial P}{\partial x} = 0 \qquad \frac{\partial P}{\partial y} = 0 \qquad \frac{\partial P}{\partial z} = -\rho g}$$

$$\frac{\partial}{\partial x} \underline{e}_x + \frac{\partial}{\partial y} \underline{e}_y + \frac{\partial}{\partial z} \underline{e}_z \to \underline{\nabla}$$

$$-\nabla P - \rho g \underline{e}_z = 0 \qquad \underline{g} = -g \underline{e}_z$$

$$-\nabla P + \rho \underline{g} = 0$$

$\boxed{\underline{\nabla} P = \rho \underline{g}}$ Static Pressure Equation!

* This is a <u>differential force balance</u>

Sometimes called a "shell balance"

* We will see several other examples! $\to$ Navier Stokes Equation