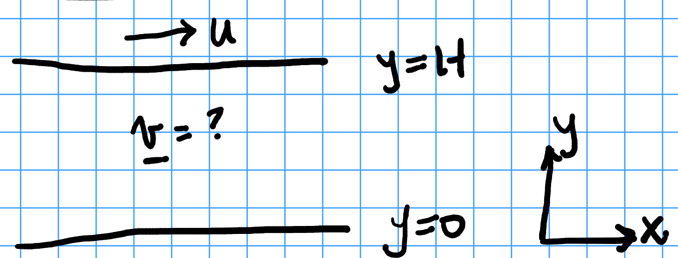
Lectures 18-19 : Couette and Poiseuille Flow

A. Couette Flow



Assume

- Flow is steady $\quad \frac{\partial}{\partial t}\underline{v} \to 0$
- Flow is 2D $\quad \frac{\partial}{\partial z} \to 0$
- Flow is unidirectional $\quad v_x(y)$ $\quad v_y = 0,\ v_z = 0$ $\quad \frac{\partial v_x}{\partial x} = 0$ ← continuity
- neglect gravity
- No ΔP

$$\rho\left(\cancel{\frac{\partial v_x}{\partial t}} + v_x \cancel{\frac{\partial v_x}{\partial x}} + \cancel{v_y \frac{\partial v_x}{\partial y}} + \cancel{v_z \frac{\partial v_x}{\partial z}}\right) = -\cancel{\frac{\partial P}{\partial x}} + \mu\left[\cancel{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2} + \cancel{\frac{\partial^2 v_x}{\partial z^2}}\right] + \underbrace{\cancel{\rho g_x}}_{0}$$

$$\underline{g} = \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}$$

$$0 = \cancel{\mu} \frac{\partial^2 v_x}{\partial y^2} \qquad v_x(0) = 0, \quad v_x(H) = u$$

$$\frac{d^2 v_x}{dy^2} = 0 \Rightarrow \int d\left(\frac{dv_x}{dy}\right) = \int 0 \, dy$$

$$\Rightarrow \frac{dv_x}{dy} = c_1 \Rightarrow \int dv_x = \int c_1 \, dy$$

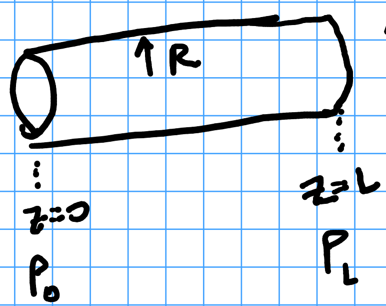
$$\Rightarrow v_x = c_1 y + c_2$$

$$v_x(0) = \cancel{c_1(0)} + c_2 = 0 \qquad c_2 = 0$$

$$v_x(H) = c_1 \cdot H = u \Rightarrow c_1 = \frac{u}{H}$$

$$\boxed{v_x(y) = \frac{u}{H} y}$$

B. Poiseuille Flow



$r \uparrow \longrightarrow z \quad \partial\theta$

$\swarrow g \quad \rightarrow g ?$

? ?

assume

- Flow is steady $\rightarrow \frac{\partial v_z}{\partial t} = 0$
- Flow is axisymmetric $\rightarrow \frac{\partial v_z}{\partial \theta}$
- Flow is unidirectional

v_z ✓ $\quad v_r = 0$, $v_\theta = 0$

By continuity Eq $\frac{\partial v_z}{\partial z} = 0$

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + v_r \cancel{\frac{\partial v_z}{\partial r}} + \frac{v_\theta}{r}\cancel{\frac{\partial v_z}{\partial \theta}} + v_z \cancel{\frac{\partial v_z}{\partial z}}\right) =$$

$$-\frac{\partial P}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right] - \rho g_z$$

$$0 = -\frac{\partial P}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right] - \rho g z$$

$$\mathcal{P} = P + \rho g z$$

$$d\mathcal{P} = dP + \rho g$$

$$0 = -\frac{\partial \mathcal{P}}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right] \checkmark$$

$v_z(R) = 0$ no slip

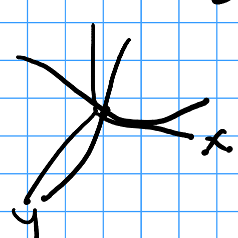
$|v_z| < \infty$

$$\frac{\partial \mathcal{P}}{\partial z} = \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right]$$

$\mathcal{P} = \mathcal{P}(z) \quad v_z = v_z(r)$

$\frac{\partial \mathcal{P}}{\partial r} = 0 \quad \frac{\partial v_z}{\partial z} = 0$

$$\frac{\partial P}{\partial z} = C = \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right]$$



$\frac{dP}{dz} = C \qquad P = Cz + b \qquad P(0) = P_0$

$P(0) = C\cdot 0 + b \qquad P(L) = P_L$

$\longrightarrow b = P_0$

$P(L) = C\cdot L + P_0 \Rightarrow C = \frac{P_L - P_0}{L}$

$C = \frac{\Delta P}{L}$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{c}{\mu} = \frac{\Delta\mathcal{P}}{\mu L}$$

$\partial \to d$ b/c $v_z = v_z(r)$ only

$$\int d\left(r\frac{dv_z}{dr}\right) = \int \frac{\Delta\mathcal{P}}{\mu L} r\, dr$$

$$r\frac{dv_z}{dr} = \frac{\Delta\mathcal{P}}{\mu L} r^2/2 + c_1$$

$$\frac{dv_z}{dr} = \frac{\Delta\mathcal{P}}{\mu L}\frac{r}{2} + \frac{c_1}{r} \Rightarrow \int dv_z = \int\left(\frac{\Delta\mathcal{P}}{\mu L}\frac{r}{2} + \frac{c_1}{r}\right)dr$$

$$v_z = \frac{\Delta\mathcal{P}}{\mu L}\frac{r^2}{4} + c_1 \ln r + c_2$$

$v_z(R) = 0$

$\to |v_z| < \infty$

$r \to 0$

0

$c_1 \ln r \to \infty$!

$c_1 = 0$

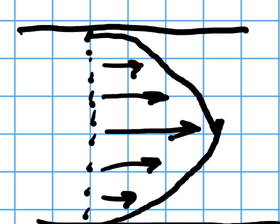
$$v_z(R) = 0 = \frac{\Delta\mathcal{P}}{\mu L}\frac{R^2}{4} + c_2 \qquad c_2 = -\frac{\Delta\mathcal{P}}{\mu L}\frac{R^2}{4}$$

$$v_z(r) = \frac{\Delta\mathcal{P}}{\mu L}\left(\frac{r^2}{4} - \frac{R^2}{4}\right)$$

$$= \frac{\Delta\mathcal{P}}{4\mu L}R^2\left(\frac{r^2}{R^2} - 1\right)$$

$\Delta\mathcal{P} < 0 \quad \mathcal{P}_L < \mathcal{P}_0$

$$\boxed{v_z(r) = -\frac{\Delta\mathcal{P}}{4\mu L}R^2\left(1 - \frac{r^2}{R^2}\right)}$$

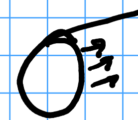


c. Friction factor

$$f = \frac{\tau_w}{\frac{1}{2}\rho u^2}$$

wall shear stress

average velocity



(i) Avg. velocity

$$Q = \int v_z \, dA = uA$$

$u \cdot A$

$\frac{1}{N}\sum_i x_i$

$$u = \frac{1}{A}\int v_z \, dA$$

$$u = \frac{1}{\pi R^2}\int_0^R \int_0^{2\pi} v_z(r)\, r \, d\theta \, dr$$

$$= \frac{2\cancel{\pi}}{\cancel{\pi} R^2}\int_0^R v_z(r)\, r\, dr$$

$$= \frac{2}{R^2}\int_0^R K\left(1 - \frac{r^2}{R^2}\right) r\, dr \qquad K = \frac{-\Delta P}{\mu L}\frac{R^2}{4}$$

$$= -\frac{\cancel{2}\,\Delta P\,\cancel{R^2}}{\cancel{R^2}\,\mu L\,\cancel{4}_{2}}\int_0^R \left(r - \frac{r^3}{R^2}\right) dr$$

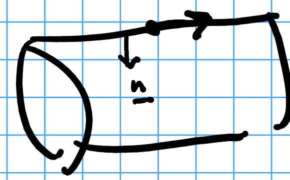
$$= \frac{-\Delta P}{2\mu L}\left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R = \frac{-\Delta P}{2\mu L}\left[\frac{R^2}{2} - \frac{\cancel{R^4}^{\,R^2}}{4\cancel{R^2}}\right]$$

$$= \frac{-\Delta P}{2\mu L}\frac{R^2}{4}$$

$$\boxed{u = \frac{-\Delta P}{8\mu L}R^2}$$

$$\boxed{v_z = 2u\left(1 - \frac{r^2}{R^2}\right)}$$

(ii) Get τ_w?



$$\underline{\underline{\tau}} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$\tau_w = \underline{\underline{\tau}}\Big|_{r=R} = \tau_{rz}\Big|_{r=R}$$

$$\tau_{rz} = \mu\left(\frac{\partial v_z}{\partial r} + \underbrace{\cancel{\frac{\partial v_r}{\partial z}}}_{0}\right)$$

$$\tau_w = \mu \frac{\partial v_z}{\partial r}\Big|_{r=R}$$

$$= \mu \frac{\partial}{\partial r}\left(2u\left(1 - \frac{r^2}{R^2}\right)\right)\Big|_{r=R}$$

$$= \mu(2u) - \frac{2r}{R^2}\Big|_{r=R} = 2\mu u\left(\frac{-2\cancel{R}}{R^{\cancel{2}}}\right)$$

$$\tau_w = \frac{-4\mu u}{R} \qquad \tau_w = \left|\tau_{rz}|_{r=R}\right| = \frac{4\mu u}{R} \qquad R = D/2$$

$$f = \frac{\tau_w}{\frac{1}{2}\rho u^2} = \frac{4\mu \cancel{u}}{R}\,\frac{2}{\rho u^{\cancel{2}}} = \frac{8\mu}{\rho u R} = \frac{16\mu}{\rho u D}$$

$$\boxed{f = \frac{16}{Re}}$$ laminar flow!