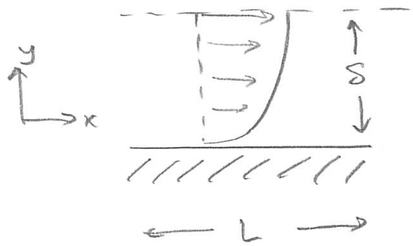
Supplemental Notes: Derivation of Boundary Layer Equations via dimensional analysis



* Governing Equations:

$$\rho \frac{D\underline{v}}{Dt} = -\underline{\nabla} P + \mu \nabla^2 \underline{v} \quad \& \quad \underline{\nabla} \cdot \underline{v} = 0$$

* Assumptions

  * 2D
  * Cartesian ← actually quite general when $\delta \ll L$ because of high radius of curvature.
  * $\delta \ll L$
  * Steady

* Scales

$\tilde{x} = x/L$ $\quad \tilde{v}_x = v_x/u$ $\quad \tilde{P} = P/\rho u^2$

$\tilde{y} = y/\delta$ $\quad \tilde{v}_y = ?$

* Continuity

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \Rightarrow \frac{u}{L}\frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \frac{1}{\delta}\frac{\partial v_y}{\partial \tilde{y}} = 0$$

$$\frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \frac{\partial}{\partial \tilde{y}}\left(\frac{L}{\delta}\frac{v_y}{u}\right) = 0 \Rightarrow \tilde{v}_y = \frac{L}{\delta}\frac{v_y}{u}$$

* x - momentum

$$\rho\left(\cancel{\frac{\partial v_x}{\partial t}} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right)$$

Steady

$$\rho(\tilde{v}_x u)\frac{\partial(\tilde{v}_x u)}{\partial(\tilde{x}L)} + \rho\left(\frac{\tilde{v}_y \delta u}{L}\right)\frac{\partial(\tilde{v}_x u)}{\partial(\tilde{y}\delta)} =$$

$$-\frac{\partial(\tilde{P}\rho u^2)}{\partial(\tilde{x}L)} + \mu\frac{\partial^2(\tilde{v}_x u)}{\partial(\tilde{x}L)^2} + \mu\cdot\frac{\partial^2(\tilde{v}_x u)}{\partial(\tilde{y}\delta)^2}$$

$$\frac{\rho u^2}{L}\tilde{v}_x\frac{\partial\tilde{v}_x}{\partial\tilde{x}} + \frac{\rho\delta u^2}{\delta L}\tilde{v}_y\frac{\partial\tilde{v}_x}{\partial\tilde{y}} = -\frac{\rho u^2}{L}\frac{\partial\tilde{P}}{\partial\tilde{x}} + \frac{\mu u}{L^2}\frac{\partial^2\tilde{v}_x}{\partial\tilde{x}^2} + \frac{\mu u}{\delta^2}\frac{\partial^2\tilde{v}_x}{\partial\tilde{y}^2}$$

divide by $\rho u^2/L$

$$\tilde{v}_x\frac{\partial\tilde{v}_x}{\partial\tilde{x}} + \tilde{v}_y\frac{\partial\tilde{v}_x}{\partial\tilde{y}} = -\frac{\partial\tilde{P}}{\partial\tilde{x}} + \underbrace{\frac{\mu u/L}{\rho u^2}}_{\frac{1}{Re_L}}\frac{\partial^2\tilde{v}_x}{\partial\tilde{x}^2} + \underbrace{\frac{\mu u/\delta^2}{\rho u^2/L}}_{\frac{1}{Re_L}\frac{L^2}{\delta^2}}\frac{\partial^2\tilde{v}_x}{\partial\tilde{y}^2}$$

* let $Re_L >> 1$

* let $\frac{1}{Re_L}\frac{L^2}{\delta^2} \approx 1 \Rightarrow \frac{\mu}{\rho u L}\frac{L^2}{\delta^2} \approx 1 \Rightarrow \frac{\mu L}{\rho u} \approx \delta^2$

$$\boxed{\delta \approx \left(\frac{\nu L}{u}\right)^{1/2}} \quad \text{or} \quad \boxed{\frac{\delta}{L} \approx \left(\frac{\nu}{uL}\right)^{1/2} = Re_L^{-1/2}}$$

$$\boxed{\tilde{v}_x\frac{\partial\tilde{v}_x}{\partial\tilde{x}} + \tilde{v}_y\frac{\partial\tilde{v}_x}{\partial\tilde{y}} = -\frac{\partial\tilde{P}}{\partial\tilde{x}} + \frac{\partial^2\tilde{v}_x}{\partial\tilde{y}^2}}$$