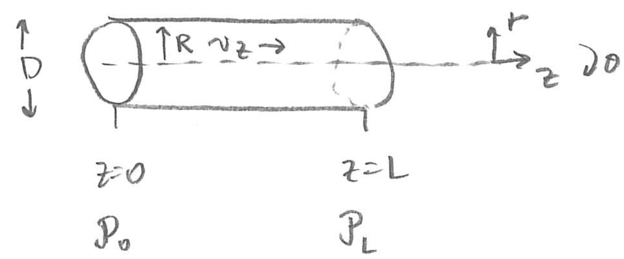
Velocity Profile near a wall in turbulent flow



$$\nabla \cdot \langle \underline{v} \rangle = 0 , \quad \nabla \cdot \underline{u} = 0$$

$$\rho \frac{D\langle \underline{v} \rangle}{Dt} = -\nabla \langle \mathcal{P} \rangle + \nabla \cdot (\langle \underline{\underline{\tau}} \rangle + \underline{\underline{\tau}}^*)$$

* unidirectional for $\langle \underline{\underline{\tau}} \rangle$ in z (b/c of average)

For the given geometry, this simplifies to:

$$0 = -\frac{\partial \langle \mathcal{P} \rangle}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left[r \left(\langle \tau_{rz} \rangle + \tau_{rz}^* \right) \right]$$

recall that: $\tau_w = -\frac{D}{4}\frac{\Delta \mathcal{P}}{L} = -\frac{D}{4}\frac{\partial \langle \mathcal{P} \rangle}{\partial z}$ (Eq. 2.3-4 on p. 35)

$$\frac{\partial \langle \mathcal{P} \rangle}{\partial z} = -\frac{4\tau_w}{D} = -\frac{2}{R}\tau_w \qquad R = D/2$$

a constant.

let $\tau_0 = -\tau_w$ because this is what the book defines.

$$\frac{2}{R}\tau_0 = \frac{1}{r}\frac{\partial}{\partial r}\left[r \left(\langle \tau_{rz} \rangle + \tau_{rz}^* \right) \right]$$

$$\frac{2r}{R}\tau_0 = \frac{\partial}{\partial r}\left[r \left(\langle \tau_{rz} \rangle + \tau_{rz}^* \right) \right]$$

separate & integrate

$$\int \frac{2r}{R}\tau_0 \, dr = \int d\left[r \left(\langle \tau_{rz} \rangle + \tau_{rz}^* \right) \right]$$

$$\frac{2r^2}{2R}\tau_0 + C = r\left(\langle \tau_{rz}\rangle + \tau_{rz}^{t}\right)$$

$$\frac{r}{R}\tau_0 + \frac{C}{r} = \langle \tau_{rz}\rangle + \tau_{rz}^{t}$$

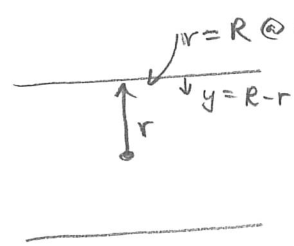
via symmetry

$$C = 0 \quad \begin{cases} \langle \tau_{rz}\rangle(0) + \tau_{rz}^{t}(0) = 0 \\ \text{or} \\ \langle \tau_{rz}\rangle(R) + \tau_{rz}^{t}(R) = \tau_0 \end{cases}$$

$$\langle \tau_{rz}\rangle + \tau_{rz}^{t} = \frac{\tau_0 r}{R}$$

← (Eq. 10.5-4), but book has a sign error.

* change of variables



$\Rightarrow r = R - y$

let $y = R - r$

(the part near the wall)

and assume $y \ll R$.

$$\langle \tau_{yz}\rangle + \tau_{yz}^{t} = \frac{\tau_0}{R}(R-y)$$

$$= \tau_0\left(1 - y/R\right)$$

← if $y \ll R$ then $1 - y/R \approx 1$

* change to look @ cartesian τ only. This is okay because $y \ll R$, so the radius of curvature is negligible. →

now we can say

$$\langle \tau_{yz}\rangle = \rho\nu \frac{d\langle v_z\rangle}{dy}$$

$$\tau_{yz}^{t} = \rho\varepsilon \frac{d\langle v_z\rangle}{dy}$$

eddy diffusivity

$$\rho(\nu+\varepsilon)\frac{\langle v_z\rangle}{dy} = \tau_0$$ ← the book makes this dimensionless.

BC: $\langle v_z\rangle(0) = 0$ (no slip @ the wall)

$$v_z^+ = \frac{\langle v_z\rangle}{(\tau_0/\rho)^{1/2}} = \frac{\langle v_z\rangle}{u_\tau}$$

$$\left(\frac{kg\,m}{s^2}\frac{1}{m^2}\;\frac{m^3}{kg}\right)^{1/2} = \left(\frac{m^2}{s^2}\right)^{1/2} = \frac{m}{s}$$

v_z^+, y^+ are called "wall variables"

$u_\tau = (\tau_0/\rho)^{1/2}$ = "friction velocity"

$y^+ = y/(\nu/u_\tau)$

$$\nu/u_\tau = \frac{m^2}{s}\;\frac{s}{m} = m$$

$$\rho(\nu+\varepsilon)\frac{u_\tau}{\nu}u_\tau\frac{dv_z^+}{dy^+} = \tau_0$$

$$\frac{u_\tau^2\rho}{\nu}(\nu+\varepsilon)\cdot\frac{1}{\tau_0}\frac{dv_z^+}{dy^+} = 1$$ ← $u_\tau^2 = \frac{\tau_0}{\rho}$

$$\frac{\cancel{\tau_0}}{\cancel{\rho}}\frac{\rho}{\nu}\cdot\frac{1}{\cancel{\tau_0}}(\nu+\varepsilon)\frac{dv_z^+}{dy^+} = 1$$

$$\boxed{\left(1+\frac{\varepsilon}{\nu}\right)\frac{dv_z^+}{dy^+} = 1\;,\quad v_z^+(0) = 0}$$

(Eq. 10.5-7)

Now it is math from here.

This is not as easy as it looks because the eddy diffusivity is not a constant

$$\frac{\varepsilon}{\nu} = (k y^+)^2\left(\frac{\partial v_z^+}{dy^+}\right)$$ ← Eq 10.4-9.

constant that is empirically fit to data.

* when $y^+ \to 0$, $\frac{\varepsilon}{\nu} \to 0$ (inner region)

$$\frac{dv_z^+}{dy^+} = 1 \quad v_z^+(0) = 0 \quad \Rightarrow \quad \boxed{v_z^+ = y^+ \quad (y^+ \to 0)}$$

* when $y^+ \to \infty$, $(Ky^+)^2 \frac{dv_z^+}{dy^+} \gg 1$ (outer region)

$$\left(Ky^+ \frac{dv_z^+}{dy^+}\right)^2 = 1$$

$$Ky^+ \frac{dv_z^+}{dy^+} = 1 \quad \Rightarrow \quad \frac{dv_z^+}{dy^+} = \frac{1}{Ky^+} \quad \Rightarrow \quad v_z^+ = \frac{1}{K} \ln y^+ + C$$

$$\boxed{v_z^+ = \frac{1}{K} \ln y^+ + C \quad (y^+ \to \infty)}$$

* C is hard to get using this approximate solution method. To get it involves a more complicated asymptotic matching technique, or, finding the full solution without approximation. Deen gives

$$C = \frac{1}{K}[\ln(4K) - 1]$$

but this doesn't fit experiment well. Instead

$$v_z^+ = 1/K \ln y^+ + C$$ is used

with $K \approx 0.4$ and $C \approx 5.5$ to fit experimental data.

Finally, lets put this back in more useful variables.

$$v_z^+ = \frac{1}{K} \ln y^+ + c \qquad K = 0.4, \; c = 5.5$$

$$v_z^+ = \frac{\langle v_z \rangle}{(\tau_0/\rho)^{1/2}} \qquad y^+ = \frac{u_\tau}{\nu} y = \left(\frac{\tau_0}{\rho}\right)^{1/2} \frac{1}{\nu}(R-r)$$

$$\langle v_z \rangle = \left(\frac{\tau_0}{\rho}\right)^{1/2} \left\{ \frac{1}{K} \ln\left[\left(\frac{\tau_0}{\rho}\right)^{1/2} \frac{R-r}{\nu} \right] + c \right\}$$

$\nu = \mu/\rho$

$$= \left(\frac{\tau_0}{\rho}\right)^{1/2} \left\{ \frac{1}{K} \ln\left[\frac{\tau_0^{1/2}}{\rho^{1/2}} \frac{\rho}{\mu} (R-r) \right] + c \right\}$$

$$\boxed{\langle v_z \rangle = \frac{\tau_0^{1/2}}{\rho^{1/2}} \left\{ \frac{1}{K} \ln\left[\frac{\tau_0^{1/2} \rho^{1/2}}{\mu} (R-r) \right] + c \right\}}$$

← in the notes p. 23-5

$K = 0.4, \; C = 5.5$

Normalized to the center line value.

$$\langle v_z \rangle (r=0) = \left(\frac{\tau_0}{\rho}\right)^{1/2} \left\{ \frac{1}{K} \ln\left[\underbrace{\frac{\tau_0^{1/2} \rho^{1/2}}{\mu} R} \right] + c \right\}$$

$$R^+ = R/(\nu/u_\tau) = \frac{u_\tau}{\nu} R = \frac{\tau_0^{1/2}}{\rho^{1/2}\nu} = \frac{\tau_0^{1/2}\rho^{1/2}}{\mu} R$$

$$\langle v_z \rangle (r=0) = \left(\frac{\tau_0}{\rho}\right)^{1/2} \left\{ \frac{1}{K} \ln R^+ + c \right\}$$

note also : $\frac{\tau_0^{1/2}\rho^{1/2}}{\mu}(R-r) = \frac{\tau_0^{1/2}\rho^{1/2}}{\mu} R\left(1 - \frac{r}{R}\right) = R^+ \frac{y}{R}$

$$\boxed{\frac{\langle v_z \rangle}{\langle v_z \rangle (r=0)} = \frac{\frac{1}{K} \ln\left[\frac{R^+ y}{R}\right] + c}{\frac{1}{K} \ln R^+ + c}}$$

Eq. 10.5-19 in book, p. 275

* lets compare this to the HW :

$$u = \frac{2}{R^2}\int_0^R \langle v_z\rangle(r)\, r\, dr$$

$$= \frac{2}{R^2}\int_0^R \left(\frac{\tau_0}{\rho}\right)^{1/2} \left\{ \frac{1}{K} \ln\left[\frac{\tau_0^{1/2}\rho^{1/2}}{\mu}(R-r)\right] + C \right\} r\, dr$$

$$= \frac{2}{R^2}\left(\frac{\tau_0}{\rho}\right)^{1/2} \int_0^R \left\{ \frac{1}{K}\ln\left[R^+\left(1-\frac{r}{R}\right)\right] + C\right\} r\, dr$$

let $\eta = 1 - r/R$ $\qquad \eta(0) = 1,\ \eta(R) = 0$

$d\eta = -\frac{1}{R}\, dr \qquad r = R(1-\eta)$

$$= \frac{2}{\cancel{R^2}}\left(\frac{\tau_0}{\rho}\right)^{1/2} \int_1^0 \left\{\frac{1}{K}\ln\left[R^+\eta\right] + C\right\} \cancel{R}(1-\eta)(-\cancel{R})\, d\eta$$

$(-1) \Rightarrow$ flip integral bounds

$$= 2\left(\frac{\tau_0}{\rho}\right)^{1/2}\left\{\int_0^1 \frac{1}{K}\ln\left[R^+\eta\right](1-\eta)\, d\eta + \int_0^1 C(1-\eta)\, d\eta\right\}$$

$$= 2\left(\frac{\tau_0}{\rho}\right)^{1/2}\left\{\int_0^1 \left(\frac{1}{K}\ln R^+ + C\right)(1-\eta)\, d\eta + \int_0^1 \frac{1}{K}\ln\eta\,(1-\eta)\, d\eta\right\}$$

$$\int_0^1 (1-\eta)\, d\eta = \eta - \frac{\eta^2}{2}\Big|_0^1 = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

mathematica

$$\int_0^1 (1-\eta)\ln\eta\, d\eta = -\eta + \frac{\eta^2}{4} + \eta\ln\eta - \frac{1}{2}\eta^2\ln\eta\Big|_0^1 = -\frac{3}{4}$$

$$u = 2\left(\frac{\tau_0}{\rho}\right)^{1/2} \left\{ \left(\frac{1}{k} \ln R^+ + c\right)\frac{1}{2} + \frac{1}{k}\left(-3/4\right) \right\}$$

$$u = \left(\frac{\tau_0}{\rho}\right)^{1/2} \left\{ \frac{1}{k} \ln R^+ + c - \frac{3}{2}\frac{1}{k} \right\}$$

$$u = \left(\frac{\tau_0}{\rho}\right)^{1/2} \left[\underset{\uparrow \atop 2.5}{\frac{1}{k}} \ln R^+ + \underbrace{c - \frac{3}{2k}}_{1.75} \right]$$

$$\boxed{u = \left(\frac{\tau_0}{\rho}\right)^{1/2} \left[2.5 \ln R^+ + 1.75 \right]}$$

similar to $v_z(r=0)$, but not quite the same.

c vs. $c - \frac{3}{2k}$

Now, finally, what is $\frac{\langle v_z \rangle}{u} = ?$

$$\frac{\langle v_z \rangle}{u} = \frac{\tau_0^{1/2}/\rho^{1/2} \left\{ \frac{1}{k} \ln\left[R^+(1 - r/R)\right] + c \right\}}{\tau_0^{1/2}/\rho^{1/2} \left\{ \frac{1}{k} \ln R^+ + c - \frac{3}{2k} \right\}}$$

$$\boxed{\frac{\langle v_z \rangle}{u} = \frac{\frac{1}{k} \ln\left[R^+(1 - r/R)\right] + c}{\frac{1}{k} \ln R^+ + c - \frac{3}{2k}}}$$

$k = 0.4$

$c = 5.5$

$R^+ = \frac{\tau_0^{1/2} \rho^{1/2}}{\mu} R$