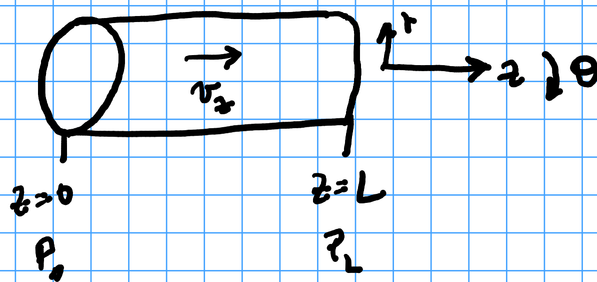
Lecture 23 – Supplement – Turbulent pipe flow



Governing Eq's:

$$\underline{\nabla}\cdot\langle\underline{v}\rangle = 0$$

$$\rho \frac{D\langle \underline{v}\rangle}{Dt} = -\underline{\nabla}\langle P\rangle + \underline{\nabla}\cdot\left(\langle\underline{\underline{\tau}}\rangle + \underline{\underline{\tau}}^*\right)$$

* For unidirectional flow, this simplifies to

$$0 = -\frac{\partial\langle P\rangle}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\langle\tau_{rz}\rangle + \tau_{rz}^*\right)\right]$$

* skipping the math, the solution is ← see posted written notes for this math

$$\langle v_z\rangle = \left(\frac{\tau_w}{\rho}\right)^{1/2}\left\{2.5 \ln\left[\frac{\sqrt{\rho\tau_w}\,(R-r)}{\mu}\right] + 5.5\right\}$$

where $\tau_w = \left|\frac{\Delta P}{L}\right| R/2$

* How find the friction factor?

(a) Calculate U (avg velocity)

(b) Get τ_w

(c) Rearrange to get f : $f = \dfrac{\tau_w}{\frac{1}{2}\rho U^2}$

(a) Get U

$$U = \frac{1}{A}\int_A \langle v_z(r)\rangle \, r\,dr\,d\theta \qquad \int_0^R\int_0^{2\pi} \dots \, d\theta \, r\,dr$$

$$= \frac{2\pi}{\pi R^2}\int_0^R \left(\frac{\tau_w}{\rho}\right)^{1/2}\left\{2.5 \ln\left[\frac{\sqrt{\rho\tau_w}\,(R-r)}{\mu}\right] + 5.5\right\} r\,dr$$

$$R^+ = \sqrt{\rho\tau_w}\,R/\mu \qquad \eta = \frac{R-r}{R} = 1 - \frac{r}{R} \qquad d\eta = -\frac{1}{R}\,dr$$

$$r = R - R\eta = R(1-\eta)$$

$$\frac{2}{R^2}\int_0^R \frac{\tau_w^{1/2}}{\rho^{1/2}}\left\{2.5\ln\left[R^+\underbrace{\left(1-\frac{r}{R}\right)}_{\eta}\right]+5.5\right\} r\,dr$$

$-R\,d\eta$

$$\frac{2}{\cancel{R^2}}\int_1^0 \frac{\tau_w^{1/2}}{\rho^{1/2}}\left\{2.5\ln\left[R^+\eta\right]+5.5\right\}(\cancel{R})(1-\eta)(-\cancel{R})\,d\eta$$

$$\frac{2\tau_w^{1/2}}{\rho^{1/2}}\int_0^1\left[2.5\ln(R^+\eta)+5.5\right](1-\eta)\,d\eta$$

$$\int_0^1 (1-\eta) = 1/2$$

$$\int_0^1 \ln\eta\,(1-\eta) = -3/4$$

$$= \frac{\cancel{2}\tau_w^{1/2}}{\rho^{1/2}}\left[\frac{1}{\cancel{2}}(2.5\ln R^+ + 5.5) + -\frac{3}{4_2}\,2.5\right]$$

$$\boxed{u = \frac{\tau_w^{1/2}}{\rho^{1/2}}\left[2.5\ln R^+ + 1.75\right]}$$

(b) already have τ_w.

(c) put together: $f = \frac{2\tau_w}{\rho u^2} \rightarrow \frac{1}{f} = \frac{\rho u^2}{2\tau_w} \Rightarrow \frac{1}{\sqrt{f}} = \frac{\sqrt{\rho}\,u}{\sqrt{2}\sqrt{\tau_w}}$

$$\frac{1}{\sqrt{f}} = \sqrt{\frac{\cancel{\rho}}{\cancel{\tau_w}}}\cdot\frac{1}{\sqrt{2}}\cdot\sqrt{\frac{\cancel{\tau_w}}{\cancel{\rho}}}\left[2.5\ln R^+ + 1.75\right]$$

$$\frac{1}{\sqrt{f}} = \frac{1}{\sqrt{2}}\left[2.5\ln R^+ + 1.75\right]$$

$$R^+ = \frac{\sqrt{\rho \tau_w} R}{\mu} = \sqrt{\frac{\rho \cdot f \rho u^2}{2}} \cdot \frac{R}{\mu} \qquad \tau_w = \frac{f \rho u^2}{2} \qquad R = D/2$$

$$= \frac{\rho u R}{\mu}\sqrt{\frac{f}{2}} = \frac{Re}{2}\sqrt{\frac{f}{2}} = \frac{Re\sqrt{f}}{2\sqrt{2}}$$

$$\log_{10} x = \frac{\ln x}{\ln 10}$$

$$\frac{1}{\sqrt{f}} = \frac{1}{\sqrt{2}}\left[2.5 \ln\left(\frac{Re\sqrt{f}}{2\sqrt{2}}\right) + 1.75\right]$$

$$\frac{1}{\sqrt{f}} = \frac{2.5}{\sqrt{2}} \ln\left(Re\sqrt{f}\right) - \frac{1}{\sqrt{2}} \ln\left(2\sqrt{2}\right) + \frac{1.75}{\sqrt{2}}$$

$$\boxed{\frac{1}{\sqrt{f}} = \underset{\uparrow\, 4.0}{4.07} \log_{10}\left(Re\sqrt{f}\right) - \underset{\uparrow\, -0.4}{0.601}} \qquad \text{BK p. 33}$$