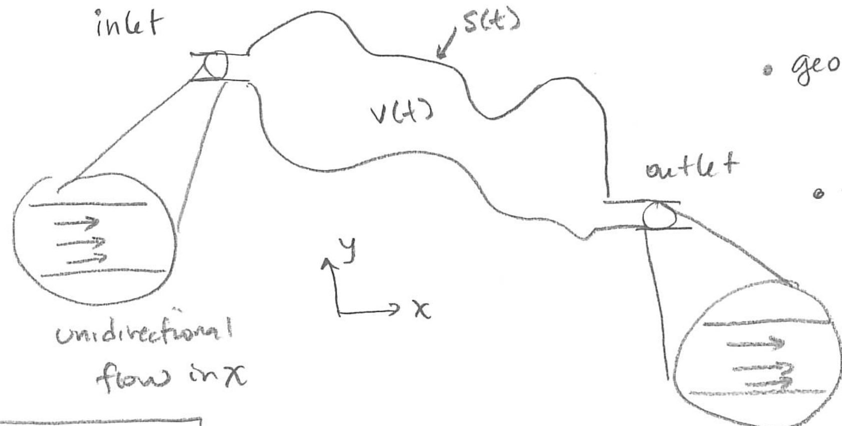
Derivation of Engineering Momentum Balance

* Integral Momentum Balance

$$\underbrace{\frac{d}{dt}\int_{V(t)} \rho \underline{v}\, dV}_{\text{accumulation}} + \underbrace{\int_{S(t)} \rho \underline{v}(\underline{v}-\underline{u})\cdot \underline{n}\, dS}_{\text{flux terms}} = \underbrace{m\underline{g} - \int_{S(t)} \underline{n} P\, dS + \int_{S(t)} \underline{n}\cdot \underline{\underline{\tau}}\, dS}_{\text{force terms}}$$



- geometry w/ a discrete inlet & outlet
- other walls are solid (no-slip/no-penetration)

$\boxed{\text{Flux Terms}}$

at inlet.

$$\int_{S_{in}} \rho \underline{v}(\underline{v}-\underline{u})\cdot \underline{n}\, dS$$

- assume fixed C.V. $\rightarrow$ $\underline{u}=0$
- unidirectional: $\underline{v} = v_x \underline{e}_x$
- constant ρ along S_{in}
- unit normal: $\underline{n} = -\underline{e}_x$

$$\Rightarrow \int_{S_{in}} \rho \underline{v}\; \underline{v}\cdot(-\underline{e}_x)\, dS$$

$$= -\rho \int_{S_{in}} v_x v_x\, dS\,(\underline{e}_x)$$

$$= -\rho S_{in} \underline{e}_x \frac{1}{S_{in}} \int v_x^2\, dS = -\rho S_{in} \langle v_x^2 \rangle \underline{e}_x$$

$$\text{let } a_{in} = \frac{\langle v_x^2\rangle}{\langle v_x\rangle^2} \Rightarrow \langle v_x^2\rangle = a_{in}\langle v_x\rangle^2$$

$$\int_{S_{in}} \rho \underline{v}\,(\underline{v}-\underline{u})\cdot\underline{n}\,dS = -\rho\, S_{in}\, a_{in} \langle v_x\rangle^2 \underline{e}_x = -w_{in}\, a_{in} \langle v_x\rangle \underline{e}_x$$

$$w = \rho\, S_{in} \langle v_x \rangle\, \underline{e}_x$$

at outlet

$$\int_{S_{out}} \rho \underline{v}\,(\underline{v}-\underline{u})\cdot\underline{n}\,dS$$

- assume fixed C.V. $\Rightarrow \underline{u}=0$
- unidirectional : $\underline{v} = v_x\, \underline{e}_x$
- constant ρ along S_{out}
- unit normal : $\underline{n} = \underline{e}_x$

$$= \rho \int_{S_{out}} \underline{v}\,\underline{v}\cdot\underline{e}_x\,dS$$

$$= \rho \int_{S_{out}} v_x \cdot v_x\, dS\, (\underline{e}_x)$$

$$= \rho\, S_{out} \cdot \frac{1}{S_{out}} \int_{S_{out}} v_x^2\, dS\, \underline{e}_x = \rho\, S_{out} \langle v_x^2 \rangle\, \underline{e}_x$$

let $a_{out} = \dfrac{\langle v_x^2\rangle}{\langle v_x\rangle^2}$

$\Rightarrow \langle v_x^2\rangle = a_{out} \langle v_x\rangle^2$

$$= \rho\, S_{out}\, a_{out} \langle v_x\rangle^2\, \underline{e}_x$$

$$w_{out} = \rho\, S_{out} \langle v_x \rangle\, \underline{e}_x$$

$$= w_{out}\, a_{out} \langle v_x\rangle\, \underline{e}_x$$

(A) No flux over the rest of $S(t) \rightarrow (\underline{v}-\underline{u}) = \underline{0}$ (no penetration condition)

$\boxed{\text{Accumulation term}}$

$$\frac{d}{dt}\int_{V(t)} \rho \underline{v}\, dV$$

- consider x-component only w/o loss of generalization.

$$\Rightarrow \frac{d}{dt}\int_{V(t)} \rho v_x \underline{e}_x\, dV = \frac{d(mv_x)_{cv}}{dt}\, \underline{e}_x$$

Force Terms

at inlet

$$\int_{S_{in}} (-\underline{n}P + \underline{n}\cdot\underline{\underline{\tau}})\, dS$$

$$= \int_{S_{in}} -(-\underline{e}_x) P \, dS$$

$$= P\underline{e}_x \int_{S_{in}} dS = P S_{in} \underline{e}_x$$

- $\underline{n} = -\underline{e}_x$
- small viscous stress at inlet, $\tau_{xi} \ll P$
- const P across S_{in}

at outlet

$$\int_{S_{out}} (-\underline{n}P + \underline{n}\cdot\underline{\underline{\tau}})\, dS$$

$$= \int_{S_{out}} -(\underline{e}_x) P \, dS$$

$$= -P S_{out} \underline{e}_x$$

- $\underline{n} = \underline{e}_x$
- small viscous stress at outlet $\tau_{xi} \ll P$
- const P across S_{out}

other surfaces

$$\int_{S_{other}} (-\underline{n}P + \underline{n}\cdot\underline{\underline{\tau}})\, dS = \underline{F}_{other}$$ ← can't simplify

Put it all together

$$\frac{d}{dt}(m v_x \underline{e}_x) + w_{out}\, a_{out} \langle v_{x,out} \rangle \underline{e}_x - w_{in}\, a_{in} \langle v_{x,in} \rangle \underline{e}_x$$

$$= m\underline{g} + P_{in} S_{in} \underline{e}_x - P_{out} S_{out} + \underline{F}_{other}$$

* simplify for x-direction only w/o loss of generality

$$\frac{d}{dt}(m v_x) = w_{in}\, a_{in} \langle v_{x,in} \rangle - w_{out}\, a_{out} \langle v_{x,out} \rangle$$
$$+ m g_x + P_{in} S_{in} - P_{out} S_{out} + F_{x,other}$$

* For multiple streams

$$\boxed{\frac{d}{dt}(m\underline{v}) = \sum_i^{in} w_i a_i \underline{v}_i - \sum_i^{out} w_i a_i \underline{v}_i + \sum_i^{in} P_i S_i - \sum_i^{out} P_i S_i + m\underline{g} + \underline{F}_{other}}$$

or

$$\boxed{\frac{d}{dt}(m\underline{v}) = \sum_i^{in} w_i a_i \underline{v}_i - \sum_i^{out} w_i a_i \underline{v}_i + \sum_i \underline{F}_i}$$