Supplemental Notes - Lecture 6

Solve a separable ODE

$$\frac{d^2z}{dt^2} = -g \qquad \left.\frac{dz}{dt}\right|_{t=0} = v_0 \qquad z(0) = z_0$$

↳ integrate

$$\frac{d}{dt}\left(\frac{dz}{dt}\right) = -g \quad \Rightarrow \quad \int d\left(\frac{dz}{dt}\right) = \int -g\, dt$$

$$\frac{dz}{dt} = -gt + c_1$$, now apply IC ; $\left.\frac{dz}{dt}\right|_{t=0} = v_0$

$$\left.\frac{dz}{dt}\right|_{t=0} = c_1 = v_0 \quad ✓ \quad \Rightarrow \quad \frac{dz}{dt} = -gt + v_0$$

↳ integrate again

$$\int dz = \int(-gt + v_0)\, dt \Rightarrow z = -\tfrac{1}{2}gt^2 + v_0 t + c_2$$

apply other IC : $z(0) = z_0$

$$z = -\tfrac{1}{2}g\cancel{(0)}^2 + v_0\cancel{(0)} + c_2 = z_0 \quad , \quad c_2 = z_0$$

$$\boxed{z = -\frac{1}{2}gt^2 + v_0 t + z_0}$$

Proof of Buckingham Pi theorem

* Let us say we know that

$$x_1 = f(x_2, x_3, \ldots, x_n)$$

where the x_i are variables, parameters, etc.

* We would like to know

$$\pi_1 = f(\pi_2, \pi_3, \ldots)$$

* We know two facts

(1) The π_i's are combinations of the x_i's

$$\pi_i = x_1^{a_1} x_2^{a_2} \ldots$$

$$\ln \pi_i = a_1 \ln x_1 + a_2 \ln x_2 + \ldots$$

(2) The x_i's are combinations of the primary dimensions, P_i (e.g. M, L, T).

$$x_1 = P_1^{b_{11}} P_2^{b_{12}} P_3^{b_{13}}$$

$$x_2 = P_1^{b_{21}} P_2^{b_{22}} P_3^{b_{23}}$$

$$x_3 = P_1^{b_{31}} P_2^{b_{32}} P_3^{b_{33}}$$

$$\ln x_1 = b_{11} \ln P_1 + b_{12} \ln P_2 + b_{13} \ln P_3 + \ldots$$

$$\ln x_2 = b_{21} \ln P_1 + b_{22} \ln P_2 + b_{23} \ln P_3 + \ldots$$

$$\ln x_3 = b_{31} \ln P_1 + b_{32} \ln P_2 + b_{33} \ln P_3 + \ldots$$

or

$$\ln \underline{x} = \underline{\underline{B}} \ln \underline{P}$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} \qquad \underline{P} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ \vdots \end{bmatrix}$$

$$\underline{\underline{B}} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots \\ b_{21} & b_{22} & b_{23} & \cdots \\ b_{31} & b_{32} & b_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

* The $\ln \pi_i$ are linear combinations of the $\ln x_i$. So, we only need to know how many linearly independent $\ln x_i$ there are.

* This is given by $n-m$, where m is the rank of the matrix $\underline{\underline{B}}$. This is the span of the basis set of $\underline{\underline{B}}$.

↳ Find the rank of the null space of $\underline{\underline{B}}$: $\underline{\underline{B}}\,\underline{x} = \underline{0}$