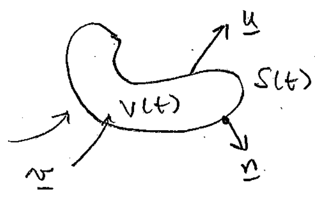
Lecture 23 - Engineering Integral Balances.

$\boxed{\text{Class Business}}$

* Review Deen. ch. 11

I. Review of Integral Balances (Handout)



control volume, V(t)

A. Mass Balance

$$\frac{d}{dt}\int_{V(t)} \rho \, dV + \int_{S(t)} \rho(\underline{v}-\underline{u})\cdot\underline{n} \, dS = 0$$

* assuming:
  - discrete inlet & outlets

$$\frac{dm}{dt} + \sum_{out} \dot{m}_i - \sum_{in} \dot{m}_i = 0$$

B. Momentum Balance

$$\frac{d}{dt}\int_{V(t)} \rho \underline{v} \, dV + \int_{S(t)} \rho \underline{v} (\underline{v}-\underline{u})\cdot\underline{n} \, dS =$$

$$m\underline{g} - \int_{S(t)} \underline{n} P dS + \int_{S(t)} \underline{n}\cdot\underline{\underline{\tau}} \, dS$$

* assuming:
  - discrete inlets and outlets
  - unidirectional flow at inlets & outlets
  - negligible viscous stress at inlets & outlets (they are non-zero, but small)
  - uniform density
  - fixed control volume

$$\frac{d}{dt}(m\underline{v}) + \sum_{out}(\dot{m}_i a_i \underline{v}_i) - \sum_{in}(\dot{m}_i a_i \underline{v}_i) = \sum_i \underline{F}_i$$

* vector equation

* $a_i = \begin{cases} 4/3 , \text{ laminar} \\ 50/49 \approx 1 , \text{ turbulent.} \end{cases}$

* v_i are average velocities over the whole opening

C. Mechanical Energy Balance

$$\frac{d}{dt}\int_{V(t)} \rho\left[\tfrac{1}{2}v^2 + gh\right] dV + \int_{S(t)} \rho\left[\tfrac{1}{2}v^2 + gh\right](\underline{v}-\underline{u})\cdot\underline{n}\, dS =$$

$$-\int_{S(t)} P(\underline{n}\cdot\underline{v})\, dS + \int_{S(t)} \underline{n}\cdot\underline{\underline{\tau}}\cdot\underline{v}\, dS + \int_{V(t)} P(\nabla\cdot\underline{v})\, dV - \int_{V(t)} \underline{\underline{\tau}} : \nabla\underline{v}\, dV$$

* assuming:
  - single inlet and outlet
  - unidirectional flow at inlet & outlet
  - negligible viscous stresses @ inlets & outlets (pipe flow = yes, contractions/expansions, fans, turbines, turbulent wake = no)
  - fixed control volume
  - uniform density
  - steady

$$\left(\frac{bv^2}{2} + \frac{P}{\rho} + gh\right)_{out} - \left(\frac{bv^2}{2} + \frac{P}{\rho} + gh\right)_{in} = \frac{1}{\dot{m}}(\underbrace{W_m}_{\text{shaft work}} - \overbrace{E_v}^{\text{friction losses}})$$

* $E_v = Q\,|\Delta \mathcal{P}|$

$$\frac{E_v}{\dot{m}} = \frac{2u^2 L}{D} f$$

II. Example problems.

* There are two types of integral balance problems.

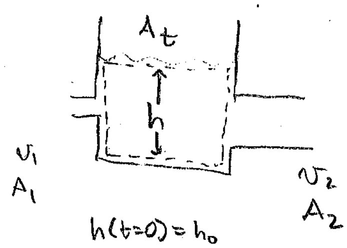
(1) Those that need the "full equations"

(2) those that need the "engineering equations"

* Choose control volumes so we get (2)! (velocities perpendicular to inlet/outlet is key.)

* Start from engineering balances unless you are forced to use the full integral balances.

A. Tank height.



$h(t=0)=h_0$

(0) Identify Balance: mass balance.

(1) Pick C.V.

(2) Assumptions: discrete inlet & outlet ✓

↓

use engineering balance

(3). Solve:

$$\frac{dm}{dt} = \dot{m}_1 - \dot{m}_2$$

$$\frac{d}{dt}(\rho A_t h) = \rho A_1 v_1 - \rho A_2 v_2$$

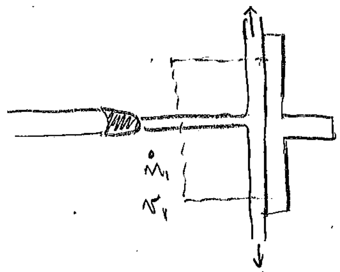
$$\frac{dh}{dt} = \frac{A_1 v_1 - A_2 v_2}{A_t}$$

$$\int dh = \int \frac{A_1 v_1 - A_2 v_2}{A_t} dt$$

$$h(t) = \frac{A_1 v_1 - A_2 v_2}{A_t} t + C \qquad h(0) = h_0, \quad C = h_0$$

$$\boxed{h(t) = \frac{A_1 v_1 - A_2 v_2}{A_t} t + h_0}$$

B. Jet striking a plate.

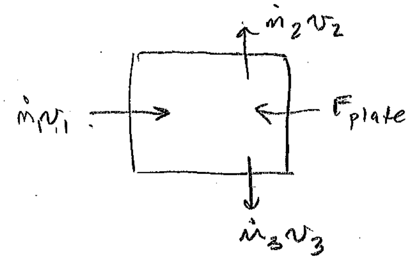


what is the force on the plate?

⓪ Identify balance: momentum balance.

① Draw C.V.

② Assumptions: ✓ satisfies engineering balance assumptions



* assume turbulent $a_i = 1$

* steady.

$$\underbrace{\frac{d}{dt}(m\underline{v})}_{0} + \sum_{out}(\dot{m}_i a_i \underline{v_i}) - \sum_{in}(\dot{m}_{in} a_i v_i) = \sum_i F_i$$

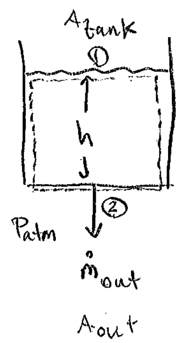
x-direction: $- \dot{m}_{in} v_{in} = - F_{plate}$.

$$\boxed{F_{plate} = \dot{m}_{in} v_{in}}$$

y-direction: $\dot{m}_2 v_2 - \dot{m}_3 v_3 = 0$

↳ not needed.

C. Velocity of draining tank



$A_{tank} >> A_{out}$.

How does $\dot{m}_{out}$ depend on height?

⓪ Identify Balance: mechanical energy

① Draw CV

② Assumptions: $\frac{dh}{dt} << \frac{\dot{m}_{out}}{A}$, quasi-steady.

↳ "fixed" C.V.

⇒ ✓ satisfies engineering balance assumptions

③ solve.

$$\frac{b v_2^2}{2} + \frac{P_2}{\rho} + g h_2 - \frac{b v_1}{2} - \frac{P_1}{\rho} - g h_1 = \frac{1}{\dot{m}} (\cancel{w_m}^{\,0} - \cancel{E_v}^{\,0})$$

$v_2 = \frac{\dot{m}_{out}}{A_{out}}$ $\quad v_1 \approx 0$ $\quad b = 1$ (turbulent) $\quad$ no pumps / turbines doing / extracting work $\quad$ no pipe flow

$P_2 = P_{atm}$ $\quad P_1 = P_{atm}$

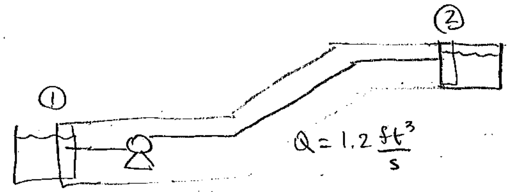
$h_2 = h$ $\quad h_1 = 0$

0, Bernoulli's equation

$$\frac{1}{2} \left(\frac{\dot{m}_{out}}{A_{out}}\right)^2 + \frac{(P_{atm} - P_{atm})}{\rho} + g h = 0$$

$$\boxed{\dot{m}_{out} = A_{out} \sqrt{2gh}}$$ Toricelli's equation

D. Pumping water up a hill



$\rho_{water} = 62.3 \frac{lbm}{ft^3}$

$\Delta h = 35 ft$

$\dot{W} = 12$ hp

$\eta = 73\%$

what are my losses due to pipe friction?

⓪ pick balance : mechanical energy

① Draw C.V.

② Assumptions : ✓ satisfies Eng. Bernoulli equation.

③. Solve.

$$\frac{\Delta v^2}{2} + \frac{\Delta P}{\rho} + g \Delta h = \frac{W_m}{\dot{m}} - \frac{E_v}{\dot{m}}$$

$v_2 = 0$ $\quad P_2 = P_{atm}$ $\quad h_2 = 35 ft$ $\quad W_m = \eta \dot{W}$ $\quad \eta = \frac{\dot{W}_{actual}}{\dot{W}_{nominal}}$

$v_1 = 0$ $\quad P_1 = P_{atm}$ $\quad h_1 = 0$

$\Delta v^2 = 0$ $\quad \Delta P = 0$ $\quad \Delta h \neq 0$

$$E_v = W_m - \dot{m} g \Delta h$$

$$\boxed{E_v = \eta \dot{W} - \rho Q g \Delta h} \qquad E_v = 0.73\,(12\,hp) - \left(62.3\,\frac{lbm}{ft^3}\right)\left(1.2\,\frac{ft^3}{s}\right) \times \left(32.2\,\frac{ft}{s^2}\right)(35\,ft)$$

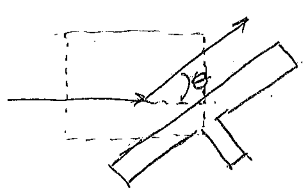
$$= 8.76\,hp - 2616.6\,\frac{lbf\,ft}{s} \times \frac{60\,s}{min} \times \frac{1\,hp}{3.3\times10^4\,\frac{lbf\,ft}{min}}$$

$$= (8.76 - 4.76)\,hp$$

$$\boxed{E_v = 4\,hp.}$$

E. Angled jet on a plate

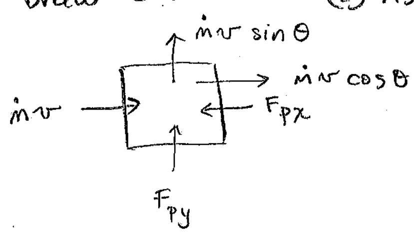
$\dot{M}$
v



What is the force on the angled plate?

⑥ Pick Balance: momentum

① Draw C.V. ② Assumptions ✓



③ Solve problem

$$\frac{d}{dt}(\dot{m} v) + \sum_{out}(\dot{m}\, a_i\, v_i) - \sum_{in}(\dot{m}\, a_i\, v_i) = \sum F_i$$

x-component: $\dot{m} v \cos\theta - \dot{m} v = -F_{px}$

$$\boxed{F_{px} = \dot{m} v (1 - \cos\theta)}$$ ← notice that sign of F_{px} works out even if we choose it wrong.

y-component: $\dot{m} v \sin\theta = F_{py}$

$$\boxed{F_{py} = \dot{m} v \sin\theta}$$