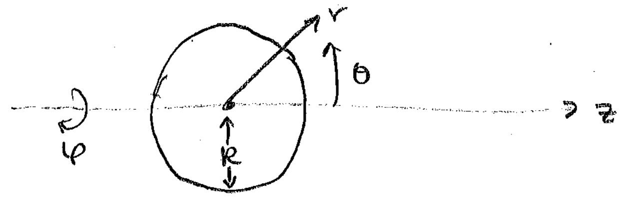
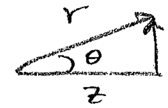
Lecture 17 Supplement — Potential Flow around a sphere





$\cos\theta = z/r$

$z = r\cos\theta$

$\underline{v} = U\underline{e}_z$ faraway

$v_r(r=R) = 0$ (no penetration)

$v_\theta(r=R) = 0$ (no slip)

$\phi = Uz = Ur\cos\theta$

$\dfrac{\partial\phi}{\partial r} = 0$

$\dfrac{\partial\phi}{\partial\theta} = 0$

* Mirrors Example 9.2-1 for a cylinder in book. & Ex. 9.2-2

potential flow $\rightarrow$ Laplace's Equation

$$\nabla^2\phi = 0 \qquad \phi = \phi(r,\theta,\varphi)$$

* ϕ is axisymmetric, so $\phi = \phi(r,\theta)$ only

$\nabla^2\phi = 0$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \underbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}}_{\text{0 because of symmetry}} = 0$$

General solution can be obtained by assuming a product solution

$$\phi(r,\theta) = f(r)\,g(\theta)$$

and using superposition.

This yields the following general solution:

$$\phi(r,\theta) = \sum_{n=0}^{\infty} \left(A_n r^n + B_n r^{-(n+1)} \right) P_n(\cos\theta)$$

where $P_n(x)$ are the Legendre Polynomials,

$$P_0 = 1$$
$$P_1 = x$$
$$P_2 = \frac{1}{2}(3x^2 - 1)$$

Now, we match the boundary conditions

$$\phi(r=\infty, \theta) = Ur\cos\theta$$

this matches $n=1$ of general solution
(don't need superposition)

$$\phi(r,\theta) = \left(A_1 r + B_1 r^{-2}\right)\cos\theta$$

(BC 1) $\phi(\infty,\theta) = Ur\cos\theta = \underbrace{A_1 r\cos\theta}_{A_1 = U} + \underbrace{B_1 r^{-2}\cos\theta}_{0 \text{ when } r\to\infty}$

(BC 2) $\left.\frac{\partial\phi}{\partial r}\right|_{r=R} = 0$

$$\frac{\partial\phi}{\partial r} = U\cos\theta - \frac{2B_1}{r^3}\cos\theta$$

$$U\cancel{\cos\theta} - \frac{2B_1}{R^3}\cancel{\cos\theta} = 0$$

$$\frac{2B_1}{R^3} = U \Rightarrow B_1 = \frac{UR^3}{2}$$

(BC 3) $\left.\frac{\partial \phi}{\partial \theta}\right|_{r=R} = 0$

$$\frac{\partial \phi}{\partial \theta} = -Ur\sin\theta - B_1 r^{-2}\sin\theta$$

$$-UR\cancel{\sin}\theta - B_1 R^{-2}\cancel{\sin}\theta = 0$$

$$B_1 R^{-2} = -UR \qquad B_1 = -UR^3$$

B_1 doesn't match between BC 2 & BC 3 $\Rightarrow$ <u>over-subscribed</u>!

- This is B/c of D'Alembert's paradox and loss of the viscosity term.
- Use BC 2, no penetration. It is "more fundamental"

$$\phi(r,\theta) = Ur\cos\theta + \frac{UR^3}{2r^2}\cos\theta$$

$$\boxed{\phi(r,\theta) = Ur\cos\theta\left(1 + \frac{R^3}{2r^3}\right)}$$

$$v_r = \frac{\partial \phi}{\partial r} = U\cos\theta - \frac{UR^3}{r^3}\cos\theta$$

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -U\sin\theta - \frac{UR^3}{2r^3}\sin\theta$$

$$\boxed{\begin{aligned} v_r &= U\cos\theta\left[1 - \frac{R^3}{r^3}\right] \\ v_\theta &= -U\sin\theta\left[1 + \frac{R^3}{r^3}\right] \end{aligned}}$$

* Sometimes this is also written in cylindrical coordinates:

cylindrical : r', θ', z'

spherical : r, θ, φ

$$z' = r\cos\theta \qquad\qquad r^2 = (r')^2 + (z')^2$$
$$r' = r\sin\theta \qquad \text{or} \qquad \cos\theta = \frac{z'}{(r'^2+z'^2)^{1/2}}$$
$$\theta' = \varphi$$

$$\boxed{\phi(r', z') = U z' \left[1 + \frac{R^3}{(r'^2 + z'^2)^{3/2}} \right]}$$

Drag on a Sphere (D'Alembert's Paradox)

To get the drag, we need the pressure. Use Bernoulli equation.

$$\frac{1}{2}v^2(r,\theta) + \frac{1}{\rho}P(r,\theta) = C \qquad v^2 = v_r^2 + v_\theta^2$$

Solve for P

$$\frac{1}{\rho}P = C - \frac{1}{2}v^2$$

$$P = \rho C - \frac{1}{2}\rho v^2 \qquad \text{when } r \to \infty,\ v^2 \to U^2,\ P \to 0$$

$$0 = \rho C - \frac{1}{2}\rho U^2$$

$$C = \frac{1}{2}U^2$$

$$P = \frac{1}{2}\rho(U^2 - v^2)$$

what is v^2?

$$v^2 = \left[U\cos\theta\left(1 - \frac{R^3}{r^3}\right)\right]^2 + \left[-U\sin\theta\left(1 + R^3/r^3\right)\right]^2$$
$$= U^2\cos^2\theta\,(1 - R^3/r^3)^2 + U^2\sin^2\theta\,(1 + R^3/r^3)^2$$

$$P = \frac{1}{2}\rho\left[U^2 - U^2\cos^2\theta(1 - R^3/r^3)^2 - U^2 \sin^2\theta\,(1 + R^3/r^3)^2\right]$$

$$= \frac{1}{2}\rho U^2\left[1 - \cos^2\theta\left(1 - \frac{2R^3}{r^3} + \frac{R^6}{r^6}\right) - \sin^2\theta\left(1 + \frac{2R^3}{r^3} + \frac{R^6}{r^6}\right)\right]$$

$$= \frac{1}{2}\rho U^2\left[\cancel{1} - \cancel{\cos^2\theta} + 2\cos^2\theta\frac{R^3}{r^3} - \cos^2\theta\frac{R^6}{r^6} - \cancel{\sin^2\theta} - 2\sin^2\theta\frac{R^3}{r^3} - \sin^2\theta\frac{R^6}{r^6}\right]$$

$\cos^2\theta + \sin^2\theta = 1$

$$= \frac{1}{2}\rho U^2\left[2\cos^2\theta\frac{R^3}{r^3} - 2\sin^2\theta\frac{R^3}{r^3} - \underbrace{(\sin^2\theta + \cos^2\theta)}_{1}\frac{R^6}{r^6}\right]$$

$$= \frac{1}{2}\rho U^2\left(\frac{R^3}{r^3}\right)\left[2\cos^2\theta - 2\sin^2\theta - \frac{R^3}{r^3}\right]$$

$2\cos^2\theta = 2 - 2\sin^2\theta$

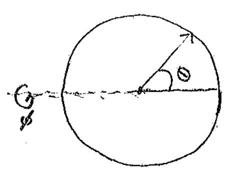
$$= \frac{1}{2}\rho U^2\left(\frac{R^3}{r^3}\right)\left[2 - 4\sin^2\theta - \frac{R^3}{r^3}\right]$$

so, $$\boxed{P(r,\theta) = \frac{1}{2}\rho U^2\left(\frac{R}{r}\right)^3\left[2 - 4\sin^2\theta - \left(\frac{R}{r}\right)^3\right]}$$

$$\boxed{P(R,\theta) = \frac{1}{2}\rho U^2\left[1 - 4\sin^2\theta\right]}$$

* Now, get the drag.

$$F_D = -\int_S \underline{e}_z \cdot \underline{n}\, P\, dS$$



$$S = \int_0^{2\pi}\int_0^{\pi} R^2 \sin\theta\, d\theta\, d\varphi$$
$$= 2\pi R^2 \int_0^{\pi} \sin\theta\, d\theta$$
$$= 2\pi R^2 (-\cos\theta)\Big|_0^{\pi}$$
$$= 2\pi R^2 (1 - -1) = 4\pi R^2 \checkmark$$

$$= -\int_0^{2\pi}\int_0^{\pi} \underline{e}_z \cdot \underline{n}\left\{\frac{1}{2}\rho U^2\left[1 - 4\sin^2\theta\right]\right\} \times R^2 \sin\theta\, d\theta\, d\varphi$$

$$= -(2\pi R^2)\left(\frac{1}{2}\rho U^2\right)\underbrace{\int_0^{\pi}(1 - 4\sin^2\theta)\sin\theta\cos\theta\, d\theta}_{\text{This integrates to 0}}$$

$$\underline{n} = \underline{e}_r = \sin\theta\cos\varphi\,\underline{e}_x + \sin\theta\sin\varphi\,\underline{e}_y + \cos\theta\,\underline{e}_z$$

$$\underline{e}_z \cdot \underline{n} = \cos\theta$$

$$\boxed{F_D = 0}$$