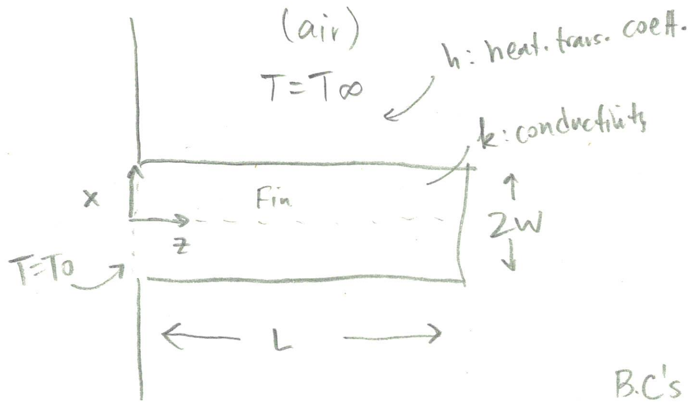
Exam Review - Fin Problem (Loss of dimensionality)



$\gamma = L/W \gg 1$

* Assume edge effects in y-dir are small.

* what is $T(x,z)$?

B.C's:

z-dir: $T(x,0) = T_0$ (west end)

$k \frac{\partial T}{\partial z} = -h\left[T(x,L) - T_\infty\right]$ (east end)

x-dir:

$k \frac{\partial T}{\partial x}(W,z) = -h\left[T(W,z) - T_\infty\right]$ (top/bottom)

$\frac{\partial T}{\partial x}(0,z) = 0$ (symmetry)

* General Balance:

$$\rho \frac{D\hat{B}}{Dt} = -\nabla \cdot \underline{f} + B_V$$

$b = \rho \hat{C}_p T$, $\hat{B} = \hat{C}_p T$

$\underline{f} = q = -k \nabla T$

$B_V = H_V$

* for const $\rho, \hat{C}_p, k$

$$\rho \hat{C}_p \frac{DT}{Dt} = k \nabla^2 T + H_V$$

* 2D (x,z), no generation $(H_V = 0)$, steady, no convection (inside fin)

$\frac{\partial T}{\partial y} = 0$ $\quad H_V = 0$ $\quad \frac{\partial T}{\partial t} = 0$ $\quad \underline{v} = 0$

$$0 = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right]$$

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} = 0}$$

* O.O.M analysis for x & z boundary conditions

x-dir $\frac{\partial T}{\partial x}(w,z) = -\frac{h}{k}\left[T(w,z) - T_\infty\right]$

$$\frac{\partial T}{\partial x} \sim \frac{T(w,z) - T(\overset{\text{center}}{0},z)}{W} \sim -\frac{h}{k}\left[T(w,z) - T_\infty\right]$$

(finite difference)

temp diff. in x-dir → $\frac{T(0,z)-T(w,z)}{T(w,z)-T_\infty} \sim \frac{hW}{k} = Bi_W$
temp diff with air →

z-dir $\frac{\partial T}{\partial z}(x,L) = -\frac{h}{k}\left[T(x,L) - T_\infty\right]$

$$\frac{\partial T}{\partial z} \sim \frac{T(x,L) - T(x,0)}{L} \sim -\frac{h}{k}\left[T(x,L) - T_\infty\right]$$

$$\frac{T(x,0) - T(x,L)}{T(x,L) - T_\infty} \sim \frac{hL}{k} = Bi_L$$

* if $W \ll L$, then $Bi_W \ll Bi_L$.

* Another way to see this:

Let $\theta = \frac{T - T_\infty}{T_0 - T_\infty}$ $\quad \zeta = z/W \quad \tilde{x} = x/W$

$$\frac{\partial T}{\partial x}(w,z) = -\frac{h}{k}\left[T(w,z) - T_\infty\right]$$

$$\frac{T_0 - T_\infty}{W}\frac{\partial \theta}{\partial \zeta} = -\frac{h}{k}\left[(T_0 - T_\infty)\,\theta(\tilde{x}=1, \zeta)\right]$$

$$\frac{\partial \theta}{\partial \zeta} = -Bi_W\, \theta\big|_{\tilde{x}=1}$$

* when $Bi_W \ll 1$, then

$$\left.\frac{\partial \theta}{\partial \xi}\right|_{\tilde{x}=0} = 0 \qquad \left.\frac{\partial \theta}{\partial \xi}\right|_{\tilde{x}=1} \approx 0$$

→ no variation in x-dir!

* But, yes variation in z-dir.

* Average across x-dir

$$\bar{T}(z) = \frac{1}{W}\int_0^W T(x,z)\,dx$$

$$\frac{1}{W}\int_0^W \frac{\partial^2 T}{\partial x^2}\,dx = \frac{1}{W}\left.\frac{\partial T}{\partial x}\right|_{x=0}^{x=W} = \frac{1}{W}\left[-\frac{h}{k}(\bar{T}-T_\infty) - 0\right]$$

↑ use B.C. $= -\frac{h}{Wk}(\bar{T}-T_\infty)$

$$\frac{1}{W}\int_0^W \frac{\partial^2 T}{\partial z^2}\,dz = \frac{d^2}{dz^2}\left[\frac{1}{W}\int_0^W T\,dz\right] = \frac{d^2\bar{T}}{dz^2}$$

* New Balance:

$$\boxed{\frac{d^2\bar{T}}{dz^2} = \frac{h}{Wk}(\bar{T}-T_\infty), \quad \bar{T}(0) = T_0, \quad \left.\frac{\partial \bar{T}}{\partial z}\right|_{z=L} = -\frac{h}{k}\left[\bar{T}(L) - T_\infty\right]}$$

* Valid for $Bi_W \ll 1$.

* Now, non-dimensionalize and then solve.

$$\theta = \frac{\bar{T}-T_\infty}{T_0 - T_\infty} \qquad \tilde{z} = Bi_W^{1/2}\,\xi = \left(\frac{hW}{k}\right)^{1/2}\cdot\frac{z}{W} = \left(\frac{h}{Wk}\right)^{1/2} z$$

* why use $Bi^{1/2}$? Technicality. Competition between heat loss in the tip & in top/bottom. Not important for now. We will talk about this more in Ch. 4. Read about in Example 3.4-2 & 3.3-2. Not on exam. (Singular Perturbation).

* Re-scaled Equations:

$$\frac{T_0 - T_\infty}{(wk/h)} \cdot \frac{d^2\theta}{d\tilde{z}^2} = \frac{h}{wk}(T_0 - T_\infty)\,\theta$$

$$\boxed{\frac{d^2\theta}{d\tilde{z}^2} - \theta = 0}$$

BC1: $\bar{T}(0) = T_0 \rightarrow (T_0 - T_\infty)\,\theta + T_\infty = T_0$

$$\boxed{\theta(0) = 1}$$

BC2: $\left.\frac{\partial \bar{T}}{\partial z}\right|_{z=L} = -\frac{h}{k}(\bar{T}(L) - T_\infty) \Rightarrow \frac{T_0 - T_\infty}{(wk/h)^{1/2}} \left.\frac{\partial \theta}{\partial \tilde{z}}\right|_{\tilde{z}=\Lambda} = -\frac{h}{k}(T_0 - T_\infty)\,\theta(\Lambda)$

let $\Lambda = L/\left(\frac{wk}{h}\right)^{1/2}$ $\Rightarrow \left.\frac{\partial \theta}{\partial \tilde{z}}\right|_{\tilde{z}=\Lambda} = -\left(\frac{wh}{k}\right)^{1/2}\theta(\Lambda)$

$$= -Bi_w^{1/2}\,\theta(\Lambda)$$

when $Bi_w \ll 1$, the RHS ≈ 0

$$\boxed{\left.\frac{\partial \theta}{\partial \tilde{z}}\right|_{\tilde{z}=\Lambda} = 0}$$

* Now, Math. Solve this one.

- Linear, constant coeff. 2^{nd} order ODE.
- characteristic EQ: $r^2 = 1 \rightarrow r = \pm 1$

$$\theta(\tilde{z}) = A \sinh(\tilde{z}) + B \cosh(\tilde{z})$$

$$\frac{d\theta}{d\tilde{z}} = A \cosh(\tilde{z}) + B \sinh(\tilde{z})$$

B.C. 1: $\theta(0) = 1 \Rightarrow \theta(0) = B \cosh(0) = B = 1$

BC 2: $\frac{\partial \theta}{\partial \tilde{z}}(\Lambda) = 0 \Rightarrow \theta = A \cosh(\Lambda) + \underbrace{B}_{1} \sinh(\Lambda)$

$$\theta = A \cosh(\Lambda) + \sinh(\Lambda)$$

$$A = -\frac{\sinh(\Lambda)}{\cosh(\Lambda)} = -\tanh(\Lambda)$$

$$\boxed{\theta(\tilde{z}) = \cosh(\tilde{z}) - \tanh(\Lambda) \sinh(\tilde{z})}$$