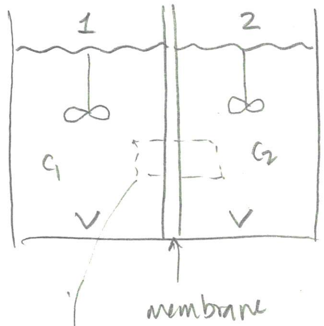
Exam Review – Membrane Diffusion (Pseudo steady problem)

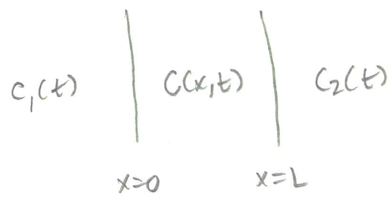


* Same volume V
* well mixed

$C_2(t) = 0$

$C_1(t) = C_0$ (at $t=0$, changed)

$\boxed{\text{* what is } C_1(t),\ C_2(t)?}$



* Area, A.

$C(x=0,t) = K\,C_1(t)$

$C(x=L,t) = K\,C_2(t)$

K: membrane partition coefficient

* General Balance :

$$\frac{\partial b}{\partial t} = -\underline{\nabla}\cdot\underline{F} + B_V$$

$b = C_A$

$\underline{F} = \underline{N}_A = C_A\underline{v}^{(M)} + \underline{J}_A^{(M)}$

$B_V = 0$

- $\underline{v}^{(M)} \approx 0$, solid membrane
- $\underline{J}_A^{(M)} = -C D_{AB} \nabla X_A$, $C \approx$ const, dilute species in a membrane

  $= -D_{AB}\nabla C_A$

- In this problem, book calls $C_A = C$, $D_{AB} = D$

$$\frac{\partial c}{\partial t} = D\nabla^2 c \quad \leftarrow \text{const Diffusivity too.}$$

- in 1 D

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}$$

* O.O.M Analysis for time.

- $\frac{\partial c}{\partial t}$ is going to change according to some process time, t_p.

$$\frac{\partial c}{\partial t} \sim \frac{KC_2 - KC_1}{t_p}$$

- Diffusion (on the RHS) is governed by the diffusion time.

$$D\frac{\partial^2 c}{\partial x^2} \sim D \cdot \frac{KC_2 - KC_1}{L^2} \sim \frac{KC_2 - KC_1}{L^2/D} \sim \frac{KC_2 - KC_1}{t_d}$$

- If $t_p \gg t_d$ then $\frac{\partial c}{\partial t}$ is negligible.

$\Rightarrow$ we only need to solve $\frac{\partial^2 c}{\partial x^2} = 0$

$$\boxed{\frac{\partial^2 c}{\partial x^2} = 0 \quad , \quad C(0,t) = KC_1(t) \quad , \quad C(L,t) = KC_2(t)}$$

* Math, solve for profile:

$$\frac{\partial^2 c}{\partial x^2} = 0 \quad \Rightarrow \quad \frac{\partial c}{\partial x} = A \quad \Rightarrow \quad C(x,t) = Ax + B$$

- BC 1: $C(0,t) = B = KC_1(t)$
- BC 2: $C(L,t) = A \cdot L + B = KC_2(t) \Rightarrow A = \frac{KC_2(t) - KC_1(t)}{L}$

$$\boxed{C(x,t) = [KC_2(t) - KC_1(t)]\frac{x}{L} + KC_1(t)}$$

* Now, solve external dynamics problem:

- Total mole balance on tanks 1 and 2

$$\frac{dn_1}{dt} = -w \qquad \frac{dn_2}{dt} = +w$$

$$w = A \cdot N_x(0,t) = A \cdot N_x(L,t)$$

Fluxes are equal, else build up in membrane.

- what is the flux?

$$N_x = -D\frac{dc}{dx} = -D \cdot \frac{KC_2(t) - KC_1(t)}{L}$$

$$= \frac{DK}{L}(c_1 - c_2) = \text{constant in } x.$$

$$= N_x(0,t) = N_x(L,t) = w/A$$

- Solve transient mole balance above:

$$n_1 = V \cdot c_1 \qquad n_2 = V \cdot c_2$$

$$V\frac{dc_1}{dt} = -A \cdot \frac{DK}{L}(c_1 - c_2) \qquad c_1(0) = c_0$$

$$V\frac{dc_2}{dt} = +A\frac{DK}{L}(c_1 - c_2) \qquad c_2(0) = 0$$

- Now, math:

$$\frac{dc_1}{dt} = -\frac{ADK}{VL}(c_1 - c_2) \qquad c_1(0) = c_0$$

$$\frac{dc_2}{dt} = \frac{ADK}{VL}(c_1 - c_2) \qquad c_2(0) = 0$$

- linear, constant coeff.
- coupled!

- solving simultaneous systems → can do by linear algebra :

$$\frac{d\underline{c}}{dt} = \begin{bmatrix} -\alpha & \alpha \\ \alpha & -\alpha \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\underline{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \qquad \alpha = \frac{ADK}{VL}$$

↳ need to do an eigenvalue problem to solve.

- easier trick : add & subtract

(adding) $\frac{d}{dt}(c_1+c_2) = \cancel{\frac{ADK}{VL}}(0) = 0$

↑ c_1+c_2 is a constant by a mole balance, it is c_0

(subtracting) $\frac{d}{dt}(c_1-c_2) = -\frac{ADK}{VL}(c_1-c_2+c_1-c_2)$

$$\frac{d}{dt}(c_1-c_2) = -\frac{2ADK}{VL}(c_1-c_2)$$

↑ new variable, γ

$$\frac{d\gamma}{dt} = -\frac{2ADK}{VL}\gamma$$

$$\gamma = \text{const.} \exp\left(-\frac{2ADK}{VL}t\right)$$

$$\gamma(0) = c_1(0) - c_2(0) = c_0 = \text{const.}$$

$$\gamma = c_0 \exp(-t/t_p)$$

$$t_p = \frac{VL}{2ADK}$$, the process time scale!

$$\frac{1}{2} \cdot \frac{V}{A} \cdot \frac{L}{D} \cdot \frac{1}{K}$$

$$\# \cdot m \cdot \frac{m}{m^2/s} \cdot \text{dimless} = s!$$

$$c_1 - c_2 = c_0 \exp(-t/t_p) \quad , \; t_p = \frac{VL}{2ADK}$$

$$c_1 + c_2 = c_0$$

$$\boxed{\begin{aligned} c_1 &= \frac{c_0}{2}\left[1 + \exp\left(t/t_p\right)\right] \\ c_2 &= \frac{c_0}{2}\left[1 - \exp\left(-t/t_p\right)\right] \end{aligned}}$$

* were we justified? How does t_p compare to t_d?

$$\frac{t_p}{t_d} = \frac{V\cancel{L}}{2A\cancel{D}K} \cdot \frac{\cancel{D}}{L^2} = \frac{V}{\underbrace{2AL}K}$$

V ← tank volume

K ← partition coeff.

$A \cdot L$ = membrane volume.

- tank volume >> membrane volume,

  so $t_p \gg t_d$. Pseudosteady is justified.

* Can do a dimensional analysis to justify once more:

$$\theta = \frac{c}{Kc_0} \; , \; \tau = t/t_p \; , \; \eta = x/L$$

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} \Rightarrow \frac{Kc_0}{t_p} \cdot \frac{\partial \theta}{\partial \tau} = D\frac{Kc_0}{L^2}\frac{\partial^2 \theta}{\partial \eta^2}$$

$$\Rightarrow \frac{1}{t_p} \cdot \frac{\partial \theta}{\partial \tau} = \underbrace{\frac{D}{L^2}}_{\frac{1}{t_d}}\frac{\partial^2 \theta}{\partial \eta^2}$$

$$\Rightarrow \underbrace{\frac{t_d}{t_p}}_{\frac{t_d}{t_p} \ll 1}\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial \eta^2}$$

$$\Rightarrow \frac{\partial^2 \theta}{\partial \eta} \approx 1$$