Exam 2

Ch En 533 – Transport Phenomena

Dec. 02, 2022 – Dec. 06, 2022

Instructions

- You can download a pdf of the exam from the Learning Suite assignment "Exam 2".
- You should submit a *single scanned pdf* of your exam answers to the Learning Suite assignment "Exam 2" before the Exam closes.
- Please name the file you submit using the following format: `Exam2_Lastname_Firstname.pdf`. You should double-check that you have submitted the correct file.
- You should start each problem on a separate piece of paper. Write neatly and clearly, and explain all assumptions and steps of your derivations. Box your answers. Unclear writing or answers will result in a loss of points.

Rules

- There is no time limit for completing the exam.
- You **may** use your notes, your book, your previous homework, and a calculator.
- You **may** use computer math tools such as Wolfram Alpha, Mathematica, Python, etc.
- You **may** use scratch paper or write on your test. However, we will only accept your submitted materials for credit.
- You **may not** communicate with anyone but the instructor or TA about the exam until after the due date has passed. This includes (but is not limited to) students in the class and graduate students outside of the class.
- You **may not** use internet searches, internet fora (e.g. stack exchange), or similar to search or request help, partial solutions, or keys for the exam problems.
- If you have any questions about the above rules, contact Dr. Tree. Violations of the rules of the exam will result in loss of points, exam failure, or course failure.

On my honor, I signify that this exam represents my work, and that I have faithfully followed the above rules to the best of my ability.

Please Sign Here ____

Contents

This exam contains:

- 3 free response questions

Problem 1. Pipeline Heat Conduction

Consider a steel pipeline ($\rho = 7832$ kg/m^3, $k = 63.9$ W/m·K, $\alpha = 18.08 \times 10^{-6}$ m^2/s) that is 1 meter in diameter and has a wall thickness of 40 mm. The pipe is heavily insulated on the outside, and before the initiation of flow, the walls of the pipe are at a uniform temperature of -20°C. With the initiation of flow, hot oil at 60°C is pumped through the pipe, creating a convective condition corresponding to $h = 500\,\mathrm{W/(m^2K)}$ at the inner surface of the pipe.

(a) Write and non-dimensionalize a transport equation (including initial and boundary conditions) that describes the heat transfer in the walls of the pipe. Assume that the walls of the pipe are thin relative to the radius of curvature, so that you may use Cartesian coordinates. Your dimensional analysis should yield a Biot number. Calculate the Biot number and comment on whether it is more appropriate to use a pseudo-steady/lumped capacitance analysis or a transient analysis.

(b) Use the FFT method to solve the resulting PDE for the time-dependent profile of the temperature distribution in the pipe wall. *Hint 1: Orient your domain so that $x = 0$ is located on the outside of the pipe and $x = L$ is located on the inside. Hint 2: You may find the following information helpful as a supplement to Table 5-2 on page 168 in Deen.*

Case	Boundary Conditions	Basis Functions
V*	$\frac{d\Phi}{dx}(0) = 0$ $\frac{d\Phi}{dx}(1) + a\Phi(1) = 0$	$\Phi_n(x) = b_n \cos(\lambda_n x)$ $b_n = \left[\frac{4\lambda_n}{2\lambda_n + \sin(2\lambda_n)}\right]^{1/2}$ $\lambda_n \tan\lambda_n = a$

*Note that there is no convenient closed form expression for the eigenvalues. They must be solved for numerically.

(c) Use the first term of your Fourier series solution to estimate the temperature of the outer pipeline wall at $t = 8$ minutes. In order to do this, you will need to numerically evaluate the eigenvalue λ_1. A Jupyter notebook has been provided to aid you in this calculation if you would like to use it.

Problem 2. Oseen Vortex

An ideal vortex is an irrotational and inviscid flow with circular streamlines of the form

$$v_\theta = \frac{\Gamma}{2\pi r} \tag{1}$$

where Γ is the "circulation" constant with units of length squared per time (e.g. m^2/s). Unfortunately, like the other irrotational and inviscid flows we have learned about, the ideal vortex is unphysical. In this case, as $r \to 0$, $v_\theta \to \infty$. Consequently, the ideal vortex does not satisfy continuity!

Therefore, we seek a vortex flow that satisfies the Navier-Stokes equation, continuity and the boundary conditions

$$v_\theta(r = 0) = 0 \tag{2}$$

$$v_\theta(r \to \infty) = \frac{\Gamma}{2\pi r} \tag{3}$$

In addition, there is one more wrinkle. It turns out that no *steady* flow can satisfy these conditions, because viscous forces at the center gradually destroy a vortex. So, we need to look at a transient problem where the initial condition is the ideal vortex

$$v_\theta(t = 0) = \frac{\Gamma}{2\pi r} \tag{4}$$

(a) Simplify the Navier-Stokes equation for a cylindrical vortex assuming there is time-dependent unidirectional flow in the θ-direction. You may also assume that there is no pressure gradient in the θ-direction.

(b) Note that there is no natural length or time scale in this problem. Therefore, perform a similarity transformation on the PDE you found in (a) and the initial and boundary conditions above. *Hint 1: Let* $f = \dfrac{v_\theta}{\Gamma/(2\pi r)}$. *Hint 2: Use what you know about the viscous penetration depth to define the variable* η.

(c) Solve the ODE you obtain in part (b) to obtain the velocity field for the "Oseen Vortex." Re-state your solution in terms of v_θ, r, and t.

Problem 3. Pulsatile Flow in a Tube

Because of mechanism of the heart, blood flow is not steady, but varies periodically in time in what is called "pulsatile flow." In this problem you are going to derive the flow profile for the transient part of pulsatile flow.

(a) Starting from the Navier-Stokes equation, derive a PDE (and initial/boundary conditions) for pulsatile flow for a fluid with density ρ and viscosity μ in a cylindrical tube of radius R assuming that the flow is unidirectional and that the pressure gradient is given by

$$\frac{\partial \mathcal{P}}{\partial z} = -a\sin(wt) \tag{5}$$

where a is an amplitude (in units of Pa/m) and w is a frequency (in units of 1/s). You may assume that the fluid is initially stagnant and that no-slip applies at the tube walls.

(b) Use appropriate length and time scales to non-dimensionalize your expression to the form

$$\frac{\partial \theta}{\partial \tau} = b\sin(\epsilon\tau) + \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial \theta}{\partial \eta}\right) \tag{6}$$

$$\theta(\eta = 1) = 0,\ \frac{\partial \theta}{\partial \eta}(\eta = 0) = 0,\ \theta(\tau = 0) = 0 \tag{7}$$

Determine expressions for the dimensionless numbers b and ϵ in terms of the variables given in part (a). For bonus points, determine the physical meaning of b and ϵ.

(c) The PDE in (b) can be solved using the FFT method. However, in this problem we are going to solve it in the low-frequency limit of small ϵ using perturbation theory.

The challenge with doing so is that the pressure gradient can be *out of phase* with the velocity. Because of this, it is easier to solve for a fictitious complex velocity that satisfies

$$\frac{\partial \Phi}{\partial \tau} = be^{i\epsilon\tau} + \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial \Phi}{\partial \eta}\right) \tag{8}$$

$$\Phi(\eta = 1) = 0,\ \frac{\partial \Phi}{\partial \eta}(\eta = 0) = 0,\ \Phi(\tau = 0) = 0 \tag{9}$$

where the original velocity is the imaginary part* of the complex velocity

$$\theta = \mathrm{Im}(\Phi) \tag{10}$$

Φ has a product solution

$$\Phi(\eta, \tau) = f(\eta)be^{i\epsilon\tau} \tag{11}$$

Verify that this is true by substituting Eq. 11 into Eq. 8 to obtain an ODE for $f(\eta)$. In addition, determine expressions for the boundary conditions for f.

(d) Use perturbation theory (up to first-order in ϵ) to solve for $f(\eta)$.

(e) Use your solution for $f(\eta)$ to find an expression for $\theta(\eta, \tau)$. For bonus points, interpret the physical meaning of the solution you obtain for θ.

*Recall that $e^{i\epsilon\tau} = \cos(\epsilon\tau) + i\sin(\epsilon\tau)$ and that $\mathrm{Im}(a + bi) = b$.