Oseen vortex

* An ideal vortex is an irrotational & inviscid flow with circular streamlines:

$$v_\theta = \frac{\Gamma}{2\pi r} \leftarrow \text{circulation, constant, units: } L^2/\text{time}$$

* Like other irrotational & inviscid flows, it has a problem. It doesn't satisfy continuity. As $r \to 0$, $v_\theta \to \infty$.

* To solve this problem, we need a viscous flow that satisfies:

$$v_\theta(r=0) = 0$$
$$v_\theta(r\to\infty) = \frac{\Gamma}{2\pi r}.$$

* It turns out that no <u>steady</u> flow can satisfy these critera. This is because viscous forces gradually destroy the vortex. So, we are interested in a transient problem where the initial condition is an ideal vortex

$$v_\theta(t=0) = \frac{\Gamma}{2\pi r}.$$

(a) Simplify the Navier-Stokes equation assuming transient, unidirectional flow $v_\theta(r,t)$.

* One can assume also that $\frac{\partial P}{\partial \theta} = 0$

* Navier-Stokes for v_θ (Table 6-8, p. 240):

$$\rho\left[\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right] =$$

$$-\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right]$$

- $v_\theta = v_\theta(r,t)$ only.
- $v_r = v_z = 0$
- $\frac{\partial P}{\partial \theta} = 0$ (given)
- $\frac{\partial v_\theta}{\partial \theta} = \frac{\partial v_\theta}{\partial z} = 0$

$$\boxed{\rho\frac{\partial v_\theta}{\partial t} = \mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)}$$

$$\boxed{\begin{aligned} &v_\theta(r=0,t) = 0 \\ &v_\theta(r\to\infty,t) = \frac{\Gamma}{2\pi r} \\ &v_\theta(r,t=0) = \frac{\Gamma}{2\pi r}\end{aligned}}$$

given

* Useful to expand the above

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r v_\theta) - \frac{1}{r^2}\frac{\partial}{\partial r}(r v_\theta)$$

(b) Transform the PDE into an ODE by a similarity transform.

* let: $f = \dfrac{v_\theta}{\Gamma/2\pi r}$ $\quad \eta = \dfrac{r}{\sqrt{\nu t}}$

* Do f first:

$$\rho \frac{\partial}{\partial t}\left(\frac{\Gamma}{2\pi r} f\right) = \frac{\mu}{r}\frac{\partial^2}{\partial r^2}\left(\frac{f\Gamma}{2\pi}\right) - \frac{\mu}{r^2}\frac{\partial}{\partial r}\left(\frac{f\Gamma}{2\pi}\right)$$

↑
r is independent of t

$$\frac{1}{\nu}\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial r^2} - \frac{1}{r}\frac{\partial f}{\partial r} \qquad \boxed{*}$$

• BC's too:

$$f(r=0, t) = 0$$
$$f(r\to\infty, t) = 1 \qquad \boxed{**}$$
$$f(r, t=0) = 1$$

* Now, do similarity transform w/ η.

$$\eta = \frac{r}{\sqrt{\nu t}}$$

$$\frac{\partial f}{\partial t} = \frac{df}{d\eta}\frac{\partial \eta}{\partial t} = \frac{df}{d\eta}\frac{\partial}{\partial t}\left(\frac{r}{\sqrt{\nu t}}\right) = \frac{df}{d\eta}\left(-\frac{1}{2}\frac{r}{(\nu t)^{3/2}}\right)\cdot \nu$$

$$= \frac{df}{d\eta}\left(-\tfrac{1}{2}\right)\frac{r}{(\nu t)^{1/2}}\frac{1}{\cancel{\nu} t}\cdot\cancel{\nu}$$

$$= \frac{df}{d\eta}\left(\frac{-\eta}{2t}\right) \quad ✓$$

$$\frac{\partial f}{\partial r} = \frac{df}{d\eta}\frac{\partial \eta}{\partial r} = \frac{df}{d\eta}\frac{\partial}{\partial r}\left(\frac{r}{\sqrt{\nu t}}\right) = \frac{df}{d\eta}\frac{1}{\sqrt{\nu t}} \quad \checkmark$$

$$\frac{\partial^2 f}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\right) = \frac{d}{d\eta}\underset{\substack{\uparrow \\ \text{from} \\ \text{above}}}{\left(\frac{df}{dr}\right)}\underset{\substack{\uparrow \\ \text{from} \\ \text{above}}}{\frac{\partial \eta}{\partial r}}$$

$$= \frac{d}{d\eta}\left(\frac{df}{d\eta}\frac{1}{\sqrt{\nu t}}\right)\frac{1}{\sqrt{\nu t}} = \frac{1}{\nu t}\frac{d^2 f}{d\eta^2} \quad \checkmark$$

* Substitute into $\boxed{*}$

$$\frac{1}{\nu}\frac{df}{d\eta}\left(\frac{-\eta}{2t}\right) = \frac{1}{\nu t}\frac{d^2 f}{d\eta^2} - \frac{1}{r}\frac{df}{d\eta}\frac{1}{\sqrt{\nu t}}$$

multiply by νt

$$-\frac{\eta}{2}\frac{df}{d\eta} = \frac{d^2 f}{d\eta^2} - \underset{\substack{\uparrow \\ 1/\eta}}{\frac{\sqrt{\nu t}}{r}}\frac{df}{d\eta}$$

$$\boxed{\frac{d^2 f}{d\eta^2} + \left(\frac{\eta}{2} - \frac{1}{\eta}\right)\frac{df}{d\eta} = 0}$$

* Transform BC's:

$$f(r=0,t) \Rightarrow \eta = \frac{0}{\sqrt{\nu t}} \Rightarrow \boxed{f(\eta=0)=0}$$

$$f(r\to\infty,t) \Rightarrow \eta = \frac{\infty}{\sqrt{\nu t}} \Rightarrow \boxed{f(\eta\to\infty)=1}$$

$$f(r,t=0) \Rightarrow \eta = \frac{r}{\sqrt{0}} \Rightarrow \boxed{f(\eta\to\infty)=1}$$

same, whew!

(c) Solve the ODE & return to dimensional variables

$$\frac{d^2f}{d\eta^2} + \left(\frac{\eta}{2} - \frac{1}{\eta}\right)\frac{df}{d\eta} = 0 \qquad f(\eta=0)=0$$
$$f(\eta\to\infty)=1$$

• let $g = \frac{df}{d\eta}$, $\frac{dg}{d\eta} = \frac{d^2f}{d\eta^2}$

$$\frac{dg}{d\eta} + \left(\frac{\eta}{2} - \frac{1}{\eta}\right) g = 0$$

separate & integrate

$$\frac{1}{g}\frac{dg}{d\eta} = \frac{1}{\eta} - \frac{\eta}{2}$$

$$\ln g = \ln \eta - \frac{\eta^2}{4} + c_1$$

$$g = e^{\ln \eta} \cdot e^{-\eta^2/4} \cdot e^{c_1} = c_1' \eta e^{-\eta^2/4}$$

$$\frac{df}{d\eta} = c_1' \eta \exp(-\eta^2/4)$$

$$f = \int c_1' \eta \exp(-\eta^2/4)\, d\eta$$

$$u = \eta^2/4 \quad du = \eta/2 \cdot d\eta$$

$$= \int c_1' \, 2\exp(-u)\, du = -2c_1' \exp(-u) + c_2$$

$$f = -2c_1' \exp(-\eta^2/4) + c_2$$

• Apply BC's

$$f(0) = -2c_1' \overset{1}{\cancel{e^0}} + c_2 = 0 \qquad c_1' = \frac{c_2}{2}$$

$$f(\infty) = -2c_1' \underset{0}{\cancel{e^{-\infty}}} + c_2 = 1 \quad \Rightarrow \quad c_2 = 1$$

$$\Rightarrow \quad c_1' = 1/2$$

- Solution in similarity vars:

$$f(\eta) = 1 - \exp(-\eta^2/4)$$

- Put in dimensional vars:

$$f = \frac{v_\theta}{\Gamma/2\pi r} \qquad \eta = \frac{r}{\sqrt{\nu t}}$$

$$\frac{v_\theta}{\Gamma/2\pi r} = 1 - \exp\left(\frac{r^2}{4\nu t}\right)$$

$$\boxed{v_\theta = \frac{\Gamma}{2\pi r}\left[1 - \exp\left(\frac{-r^2}{4\nu t}\right)\right]}$$