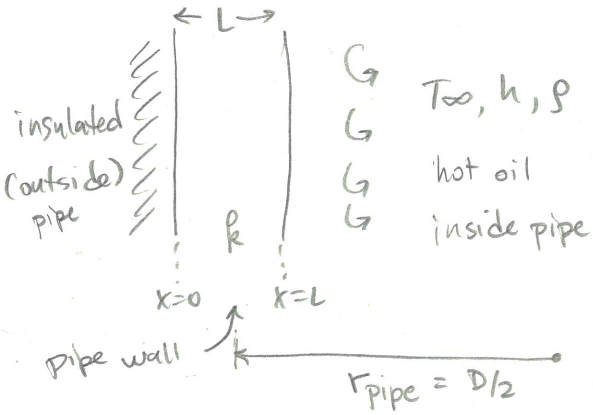
Heat Transfer through the walls of a pipe



$L = 0.04\,m$

$h = 500\ W/m^2K$

$k = 63.9\ W/mK$

$\alpha = 18.8 \times 10^{-16}\ m^2/s$

$T_\infty = 60°C = 333K$

$T(t=0) = T_0 = -20°C = 253\,K$

(a) write dimensionless transport equations (and ICs, BCs)

* Heat equation : constant ρ, $\hat{C}_p$, k

$$\rho \hat{C}_p \frac{DT}{Dt} = k \nabla^2 T + H_v$$

- No velocity (solid wall), no generation

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

- 1D (x-dir), Cartesian

$$\boxed{\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}}$$

* Initial & Boundary Conditions

$$\boxed{T(x, t=0) = T_0 = 253K}$$ initial condition

$$\boxed{\frac{dT}{dx}(x=0, t) = 0}$$ Left BC. Insulated / no flux

$$\boxed{-k \frac{dT}{dx}(x=L, t) = h\left[T(x=L, t) - T_\infty\right]}$$ Right BC convection

* Non-dimensionalize:

- let $\theta = \dfrac{T - T_\infty}{T_0 - T_\infty}$, $\eta = x/L$, $\tau = \dfrac{t\alpha}{L^2}$ ← $\dfrac{L^2}{\alpha}$: diffusion time

$$\theta = \frac{T - 333k}{-80k} , \quad \eta = \frac{x}{0.04m} , \quad \tau = t/851s$$

- substitute into heat equation

$$\frac{\cancel{T_0 - T_\infty}}{\cancel{L^2/\alpha}} \frac{\partial \theta}{\partial \tau} = \cancel{\alpha} \frac{\cancel{T_0 - T_\infty}}{\cancel{L^2}} \frac{\partial^2 \theta}{\partial \eta^2}$$

$$\boxed{\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial \eta^2}}$$

- substitute into IC & BC's

$$\theta(\eta, \tau=0) = \frac{T_0 - T_\infty}{T_0 - T_\infty} = 1$$

$$\boxed{\theta(\eta, \tau=0) = 1} \quad (IC)$$

$$\boxed{\frac{\partial \theta}{\partial \eta}(\eta=0, \tau) = 0} \quad (\text{left BC})$$

$$-k \frac{(\cancel{T_0 - T_\infty})}{L} \frac{\partial \theta}{\partial \eta}(\eta=1, t) = h\left[(\cancel{T_0 - T_\infty})\, \theta(\eta=1, t) + \cancel{T_\infty} - \cancel{T_\infty}\right]$$

$$\frac{\partial \theta}{\partial \eta}(\eta=1, t) = -\frac{hL}{k} \theta(\eta=1, t)$$

↖ Biot #

$$\boxed{\frac{\partial \theta}{\partial \eta}(\eta=1, t) + Bi\, \theta(\eta=1, t) = 0}$$

* The Biot number:

$$Bi = \frac{hL}{k} = \frac{500\ W/m^2K \cdot 0.04m}{63.9\ W/mK} = 0.313$$

$$\boxed{Bi = 0.313}$$

For a pseudosteady / lumped capacitance model to be valid Bi must be much less than 1 ($Bi \ll 1$). this is not the case here, so a transient model is more appropriate

(b) use the FFT model to solve the PDE

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial \eta^2} \qquad \theta(\eta, \tau=0) = 1$$

$$\frac{\partial \theta}{\partial \eta}(\eta=0, t) = 0$$

$$\frac{\partial \theta}{\partial \eta}(\eta=1, t) + Bi\,\theta(\eta=1, t) = 0$$

* Identify Basis functions

- The eigen problem is not in the book. See hints in the problem. I do it explicitly here.
- The homogeneous boundaries are along η. The η-dir is the eigenvalue problem.

$$\mathcal{L}\psi = \Lambda\psi$$

$$\downarrow$$

$$\frac{d^2\psi}{d\eta^2} = -\lambda^2\psi \qquad \frac{d\psi}{d\eta}(0) = 0$$

$$\frac{d\psi}{d\eta}(1) + Bi\,\psi(1) = 0$$

- For $-\lambda^2$, we have sines and cosines.

$$\phi(y) = a\sin(\lambda y) + b\cos(\lambda y)$$

$$\frac{d\phi}{dy} = a\lambda\cos(\lambda y) - b\lambda\sin(\lambda y)$$

- Apply BC's:

$$\frac{d\phi}{dy}(0) = a\lambda = 0 \quad \Rightarrow \quad a = 0$$

$$\frac{d\phi}{dy}(1) + Bi\,\phi(1) = -\cancel{b}\lambda\sin(\lambda) + Bi\,\cancel{b}\cos\lambda = 0$$

$$-\frac{\lambda\sin\lambda}{\cos\lambda} + Bi = 0$$

$\lambda\tan\lambda = Bi$, solve for eigenvalues

- Normalize the eigenfunctions:

$$\int_0^1 b\cos(\lambda_n y)\, b\cos(\lambda_n y)\, dy = 1$$

$$b^2\int_0^1 \cos^2(\lambda_n y)\, dy = 1$$

$$b^2\left[\frac{\sin(2\lambda_n y)}{4\lambda_n} + \frac{y}{2}\right]_0^1 = 1$$

$$b^2\left[\frac{\sin(2\lambda_n)}{4\lambda_n} + \frac{1}{2}\right] = 1$$

$$b^2\left[\frac{\sin(2\lambda_n) + 2\lambda_n}{4\lambda_n}\right] = 1$$

$$b = \left[\frac{4\lambda_n}{2\lambda_n + \sin(2\lambda_n)}\right]^{1/2}$$

- In summary

$$\boxed{\begin{array}{lll} \psi_n(\eta) = b_n \cos(\lambda_n \eta) & & \\ \text{where} & \lambda_n \tan \lambda_n = Bi & \text{for } n = 1, 2, \ldots \\ \text{and} & b_n = \left[\dfrac{4\lambda_n}{2\lambda_n + \sin(2\lambda_n)} \right]^{1/2} & \end{array}}$$

* Define the FFT and take FFT of the PDE

$$c_n(\tau) = \int_0^1 \theta(\eta, \tau)\, \psi_n(\eta)\, d\eta$$

$$\theta(\eta, \tau) = \sum_{n=1}^{\infty} c_n(\tau)\, \psi_n(\eta)$$

- FFT of LHS

$$\int_0^1 \frac{\partial \theta}{\partial \tau} \psi_n(\eta)\, d\eta = \frac{\partial}{\partial \tau} \int_0^1 \theta \cdot \psi_n\, d\eta = \frac{\partial c_n}{\partial \tau}$$

- FFT of RHS

$$\int_0^1 \frac{\partial^2 \theta}{\partial \eta^2} \psi_n\, d\eta \longrightarrow$$ integration by parts $\int u\, dv = uv - \int v\, du$

$$= \psi_n \frac{d\theta}{d\eta}\Big|_0^1 - \int_0^1 \frac{d\theta}{d\eta} \frac{d\psi}{d\eta}\, d\eta \quad \left\{ \begin{array}{ll} u = \psi_n & v = \dfrac{d\theta}{d\eta} \\ du = \dfrac{d\psi}{d\eta} d\eta & dv = \dfrac{d^2\theta}{d\eta^2} d\eta \end{array} \right.$$

$$= \psi_n(1) \frac{d\theta}{d\eta}(1) - \psi_n(0) \cancel{\frac{d\theta}{d\eta}(0)} - \int_0^1 \frac{d\theta}{d\eta} \frac{d\psi}{d\eta}\, d\eta$$

not zero. wait & see

0 (from BC)

$$u = \frac{d\psi}{d\eta} \qquad v = \theta$$

$$du = \frac{d^2\psi}{d\eta^2} d\eta \qquad dv = \frac{d\theta}{d\eta} d\eta$$

$$= \psi_n(1)\frac{d\theta}{d\eta}(1) - \left[\frac{d\psi}{d\eta}\theta\Big|_0^1 - \int_0^1 \theta \frac{d^2\psi}{d\eta^2}\, d\eta \right]$$

$$= \psi_n(1)\frac{d\theta}{d\eta}(1) - \frac{d\psi}{d\eta}(1)\,\theta(1) + \underbrace{\frac{d\psi}{d\eta}(0)\,\theta(0)}_{0 \text{ (see below)}} + \int_0^1 \theta \frac{d^2\psi}{d\eta^2}\, d\eta$$

$$\left[\begin{array}{l} \psi_n = b_n \cos(\lambda_n \eta) \\ \dfrac{d\psi_n}{d\eta} = -b_n \lambda_n \sin(\lambda_n \eta) \\ \dfrac{d^2\psi_n}{d\eta^2} = -b_n \lambda_n^2 \cos(\lambda_n \eta) \\ \qquad = -\lambda_n^2 \psi_n \end{array}\right\} \begin{array}{l} \psi_n(1) = b_n \cos(\lambda_n) \\ \dfrac{d\psi_n}{d\eta}(1) = -b_n \lambda_n \sin(\lambda_n) \\ \dfrac{d\psi}{d\eta}(0) = 0 \end{array}$$

$$= b_n \cos\lambda_n \frac{d\theta}{d\eta}(1) + b_n \lambda_n \sin\lambda_n\, \theta(1) + \int_0^1 -b_n\lambda_n^2 \psi_n \theta\, d\eta$$

$$\frac{d\theta}{d\eta}(1) = -Bi\,\theta(1), \quad \lambda_n \tan\lambda_n = Bi$$

$$b_n \cos\lambda_n(-Bi\,\theta(1)) + b_n \lambda_n \sin\lambda_n\, \theta(1)$$

$$\lambda_n \tan\lambda_n = \lambda_n \frac{\sin\lambda_n}{\cos\lambda_n}$$

$$\lambda_n \sin\lambda_n = \cos\lambda_n\, Bi$$

$$- b_n \cos\cancel{\lambda_n}\, Bi\, \theta(1) + b_n \cos\cancel{\lambda_n}\, Bi\, \theta(1)$$

both cancel! is zero!

$$= \int_0^1 -\lambda_n^2 \psi_n \theta\, d\eta = -\lambda_n^2 \int_0^1 \theta \psi_n\, d\eta$$

$$= \lambda_n^2 c_n$$

- ODE from FFT:

$$\boxed{\frac{dc_n}{d\tau} = -\lambda_n^2 c_n}$$

- FFT of initial condition

$$\int_0^1 \theta(\tau=0)\,\Psi_n \, dy = \int_0^1 \Psi_n \, dy$$

$$= \int_0^1 b_n \cos(\lambda_n y)\, dy$$

$$= \frac{b_n}{\lambda_n} \sin(\lambda_n y) \Big|_0^1$$

$$\boxed{C_n(0) = \frac{b_n}{\lambda_n} \sin \lambda_n}$$

* Solve the transformed equation

$$\frac{dc_n}{d\tau} = -\lambda_n^2 c_n \qquad c_n(0) = \frac{b_n}{\lambda_n} \sin \lambda_n$$

$\downarrow$

separate & integrate

$$C_n(\tau) = C_n(0) \exp(-\lambda_n^2 \tau)$$

$$\boxed{C_n(\tau) = \frac{b_n}{\lambda_n} \sin \lambda_n \exp(-\lambda_n^2 \tau)}$$

* Now, put into Fourier series

$$\theta(y,\tau) = \sum_{n=1}^{\infty} C_n(\tau)\, \Psi_n(y)$$

$$= \sum_{n=1}^{\infty} \frac{b_n \sin \lambda_n}{\lambda_n} \exp(-\lambda_n^2 \tau) \cdot b_n \cos(\lambda_n y)$$

$$= \sum_{n=1}^{\infty} b_n^2 \frac{\sin \lambda_n}{\lambda_n} \exp(-\lambda_n^2 \tau) \cos(\lambda_n y)$$

recall $b_n^2 = \dfrac{4 \lambda_n}{2\lambda_n + \sin(2\lambda_n)}$

so $b_n^2 \cdot \dfrac{\sin \lambda_n}{\lambda_n} = \dfrac{4 \sin \lambda_n}{2\lambda_n + \sin(2\lambda_n)}$

Our final solution is then:

$$\boxed{\theta(\eta,\tau) = \sum_{n=1}^{\infty} \frac{4 \sin \lambda_n}{2\lambda_n + \sin(2\lambda_n)} \exp(-\lambda_n^2 \tau) \cos(\lambda_n \eta) \qquad \lambda_n \tan \lambda_n = Bi}$$

(c) Calculate the temperature at the surface of the pipe at t = 8 min using first term of series.

$$\theta(\eta,\tau) = \frac{T - 333\,K}{-80\,K} = \frac{4 \sin \lambda_1}{2\lambda_1 + \sin(2\lambda_1)} \exp(-\lambda_1^2 \tau) \overset{1}{\cancel{\cos(0)}}$$

$$\tau = \frac{8 \cdot 60\,s}{85.1\,s} = 5.64$$

$\eta = 0$, $Bi = 0.313$ $\rightarrow$ solve for λ_1 using numerical methods. $\lambda_1 = 0.53189$

$$\frac{T - 333\,K}{-80\,K} = 1.047 \times 0.2028$$

$$\boxed{T = 316\,K = 43\,°C}$$