Pulsatile Flow in a Tube

(a) Obtain a PDE for pulsatile flow assuming unidirectional flow and sinusoidal pressure gradient

Unidirectional flow: $v_z = v_z(r,t)$ $\quad v_r = v_\theta = 0$

sinusoidal pressure gradient: $\dfrac{\partial \mathcal{P}}{\partial z} = -\underset{\uparrow}{a} \sin \underset{\uparrow}{\omega} t$

amplitude units: $\frac{Pa}{m} = \frac{kg}{m^2 s^2}$ $\quad$ frequency units: $1/s$

- Navier stokes for v_z in cylindrical coordinates (Table 6-8 in Deen)

$$\rho\left[\frac{\partial v_z}{\partial t} + \cancel{v_r \frac{\partial v_z}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}} + \cancel{v_z \frac{\partial v_z}{\partial z}}\right] =$$

$$-\frac{\partial \mathcal{P}}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right]$$

$$v_r = 0,\ v_\theta = 0;\ \frac{\partial v_z}{\partial z} = \frac{\partial^2 v_z}{\partial z^2} = \frac{\partial^2 v_z}{\partial \theta^2} = 0$$

$$\frac{\partial v_z}{\partial t} = -\frac{1}{\rho}\frac{\partial \mathcal{P}}{\partial z} + \frac{\mu}{\rho}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)$$

$$\boxed{\frac{\partial v_z}{\partial t} = \frac{a}{\rho}\sin(\omega t) + \frac{\mu}{\rho}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)}$$

Boundary conditions:

$\boxed{v_z(r=R) = 0}$ (no slip) $\quad \boxed{\frac{\partial v_z}{\partial r}(r=0) = 0}$ (symmetry) $\quad \boxed{v_z(t=0) = 0}$ initial stagnant.

(b) Non-dimensionalize using

$$\eta = r/R \quad , \quad \tau = \frac{\nu t}{R^2} = \frac{\mu t}{\rho R^2}$$

viscous diffusion time scale: R^2/ν

$$\theta = \frac{v_z R}{\nu}$$

$$\frac{\nu}{R}\frac{\nu}{R^2}\frac{\partial \theta}{\partial \tau} = \frac{a}{\rho}\sin\left(\omega \frac{R^2}{\nu}\tau\right) + \frac{\nu}{R^2}\frac{1}{\eta}\frac{\partial}{\partial \eta}\left[\cancel{R}\eta \cdot \frac{1}{\cancel{R}}\frac{\partial}{\partial \eta}\left(\frac{\theta \nu}{R}\right)\right]$$

$$\frac{\nu^2}{R^3}\frac{\partial \theta}{\partial \tau} = \frac{a}{\rho}\sin\left(\omega R^2/\nu \; \tau\right) + \frac{\nu^2}{R^3}\frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial \theta}{\partial \eta}\right)$$

$$\frac{\partial \theta}{\partial \tau} = \frac{R^3 a}{\rho \nu^2}\sin\left(\frac{\omega R^2}{\nu}\tau\right) + \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial \theta}{\partial \eta}\right)$$

*up to 3 bonus points

let $\boxed{\varepsilon = \omega R^2/\nu} = R^2/\nu / 1/\omega = \dfrac{\text{② viscous diffusion time}}{\text{pressure time const.}}$

$\|$

① Strouhal number, St

let $\boxed{b = \dfrac{R^3 a}{\rho \nu^2}} = \dfrac{R^2}{\mu(\nu/R)} \cdot a = \dfrac{R^2}{\mu u} a = \dfrac{\text{③ pressure gradient}}{\text{viscous stress gradient}}$

$$\boxed{\frac{\partial \theta}{\partial \tau} = b \sin(\varepsilon \tau) + \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial \theta}{\partial \eta}\right)}$$

$$\boxed{\varepsilon = \omega R^2/\nu} \quad \boxed{b = \frac{R^3 a}{\rho \nu^2}}$$

Boundary & initial conditions

$\boxed{\theta(\eta = 1) = 0}$ (no slip)

$\boxed{\frac{\partial \theta}{\partial \eta}(\eta = 0) = 0}$ (symmetry)

$\boxed{\theta(\tau = 0) = 0}$ (stagnant initially)

(c) verify that $\Phi = f(\eta)\, b e^{i\varepsilon\tau}$ for the PDE

$$\frac{\partial \Phi}{\partial \tau} = b e^{i\varepsilon\tau} + \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial \Phi}{\partial \eta}\right)$$

and find an ODE for $f(\eta)$.

$$\frac{\partial}{\partial \tau}\left(f(\eta)\, b e^{i\varepsilon\tau}\right) = b e^{i\varepsilon\tau} + \frac{1}{\eta}\frac{\partial}{\partial \eta}\left[\eta \frac{\partial}{\partial \eta}\left(f(\eta)\, b e^{i\varepsilon\tau}\right)\right]$$

$$i\varepsilon\, \cancel{b e^{i\varepsilon\tau}}\, f = \cancel{b e^{i\varepsilon\tau}} + \cancel{b e^{i\varepsilon\tau}}\, \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial f}{\partial \eta}\right)$$

cancels, is a solution!

$$i\varepsilon f = 1 + \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial f}{\partial \eta}\right)$$

or

$$\boxed{\frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial f}{\partial \eta}\right) - i\varepsilon f + 1 = 0}$$

probably more correct to use total derivatives now. Either ok for Exam.

Boundary conditions

$$\boxed{f(\eta = 1) = 0} \qquad \boxed{\frac{\partial f}{\partial \eta}(\eta = 0) = 0}$$

* Can also expand derivatives:

$$\frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial f}{\partial \eta}\right) = \frac{1}{\eta}\frac{\partial f}{\partial \eta} + \frac{\partial^2 f}{\partial \eta^2}$$

$$\boxed{\frac{d^2 f}{d\eta^2} + \frac{1}{\eta}\frac{df}{d\eta} - i\varepsilon f + 1 = 0}$$

(d) Solve Expression from (c) using perturbation theory.

$$f(\eta) = f_0 + \varepsilon f_1 + O(\varepsilon^2)$$

* Substitute into ODE:

$$\frac{d^2}{d\eta^2}(f_0 + \varepsilon f_1) + \frac{1}{\eta}\frac{d}{d\eta}(f_0 + \varepsilon f_1) - i\varepsilon(f_0 + \varepsilon f_1) + 1 = 0$$

at order 0: $\dfrac{d^2 f_0}{d\eta^2} + \dfrac{1}{\eta}\dfrac{df_0}{d\eta} + 1 = 0$ ← no $\underline{i\varepsilon f_0}$

at order 1: $\varepsilon\dfrac{d^2 f_1}{d\eta^2} + \varepsilon\dfrac{1}{\eta}\dfrac{df_1}{d\eta} - i\varepsilon f_0 = 0$ ← $\underline{\text{no } 1}$

* Boundary conditions:

$$f_0(\eta=1) + \varepsilon f_1(\eta=1) = 0 \Rightarrow f_0(\eta=1) = 0, \quad f_1(\eta=1) = 0$$

$$\frac{df_0}{d\eta}(\eta=0) + \varepsilon\frac{df_1}{d\eta}(\eta=1) = 0 \Rightarrow \frac{df_0}{d\eta}(\eta=0) = 0, \quad \frac{df_1}{d\eta}(\eta=0) = 0$$

* Solve at $O(\varepsilon^0)$:

$$\frac{d^2 f_0}{d\eta^2} + \frac{1}{\eta}\frac{df_0}{d\eta} = -1$$

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{df_0}{d\eta}\right) = -1 \quad \text{integrate}$$

$$\eta\frac{df_0}{d\eta} = -\frac{\eta^2}{2} + c_1$$

$$\frac{df_0}{d\eta} = -\frac{\eta}{2} + \frac{c_1}{\eta} \quad \text{integrate}$$

$$f_0 = -\frac{\eta^2}{4} + c_1 \ln \eta + c_2$$

- apply BC's:

$\frac{df_0}{d\eta}(\eta=0)$ diverges as $\eta \to 0$, so $c_1 = 0$

$f_0(\eta=1) = -\frac{1}{4} + c_2 = 0 \Rightarrow c_2 = 1/4$

$$\boxed{f_0 = \frac{1}{4}(1-\eta^2)}$$

* Solve at $O(\varepsilon^1)$:

$$\frac{d^2 f_1}{d\eta^2} + \frac{1}{\eta}\frac{df_1}{d\eta} - i f_0 = 0$$

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta \frac{df_1}{d\eta}\right) = \frac{i}{4}(1-\eta^2)$$

$$\frac{d}{d\eta}\left(\eta \frac{df_1}{d\eta}\right) = \frac{i}{4}(\eta - \eta^3)$$

integrate

$$\eta \frac{df_1}{d\eta} = \frac{i}{4}\left(\frac{\eta^2}{2} - \frac{\eta^4}{4}\right) + c_1$$

$$\frac{df_1}{d\eta} = \frac{i}{4}\left(\frac{\eta}{2} - \frac{\eta^3}{4}\right) + \frac{c_1}{\eta}$$

integrate

$$f_1 = \frac{i}{4}\left(\frac{\eta^2}{4} - \frac{\eta^4}{16}\right) + c_1 \ln \eta + c_2$$

- Apply BC's

$\frac{df_1}{d\eta}(\eta=0)$ diverges as $\eta \to 0$, so $c_1 = 0$

$f_1(\eta=1) = \frac{i}{4}\left(\frac{1}{4} - \frac{1}{16}\right) + c_2 = 0$

$c_2 = -\frac{3i}{64}$

$$\boxed{f_1 = \frac{i}{64}(4\eta^2 - \eta^4 - 3)}$$

* Put it all together:

$$\boxed{f(\eta) = \frac{1}{4}(1-\eta^2) + \frac{i\varepsilon}{64}(4\eta^2 - \eta^4 - 3) + O(\varepsilon^2)}$$

(e) Find an expression for $\theta(\eta, \tau)$:

$$\Phi = f(\eta)\, b e^{i\varepsilon\tau}$$

$$\Phi = \frac{b e^{i\varepsilon\tau}}{4}(1-\eta^2) + \frac{i\varepsilon b e^{i\varepsilon\tau}}{64}(4\eta^2 - \eta^4 - 3) + O(\varepsilon^2)$$

$$e^{i\varepsilon\tau} = \cos(\varepsilon\tau) + i\sin(\varepsilon\tau)$$

$$i\varepsilon e^{i\varepsilon\tau} = i\varepsilon\cos(\varepsilon\tau) + i^2\varepsilon\sin(\varepsilon\tau)$$

$$= i\varepsilon\cos(\varepsilon\tau) - \varepsilon\sin(\varepsilon\tau)$$

$$\theta = \mathrm{Im}(\Phi)$$

$$\boxed{\theta = \frac{b\sin(\varepsilon\tau)}{4}(1-\eta^2) + \frac{b\varepsilon\cos(\varepsilon\tau)}{64}(4\eta^2 - \eta^4 - 3) + O(\varepsilon^2)}$$

* The zero order term is in phase. ①   * up to 3 bonus points

The first order term is out of phase ②

* The zero order term is traditional poiseuille ③
flow, but modulated by $b\sin(\varepsilon\tau)$.