Introduction

The Authors S.P. Burke and T.E.W. Schumann first explained the two classes of flames which are first the Bunsen type in which the combustible gas and air are premixed before ignition occurs and flames in which the combustible gas and air meet. The later class of flames the authors applied the term diffusion flames. Diffusion flames have several applications, that is in industrial furnaces and in flares to name a few. In their paper they made an analysis of the diffusion flame and came up with a mathematical presentation of the theory and compared it to the experimental results they got. Types of diffusion flames include the flame of match, of a candle and of the familiar gas-jet burner. According to the authors in 1928 they found out that many investigations had been made on the premixed type of flames which included an adequate theory that has been advanced to account for the shape and general properties of the characteristic Bunsen cone however during that time diffusion flames had received a scant attention. The authors used the apparatus shown in Figure 1 which show two concentric tubes which were used to come up with the mathematical analysis.

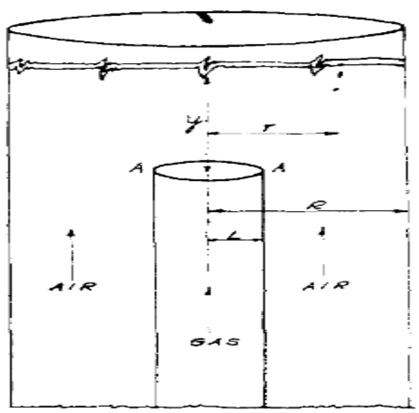


Figure 1-Diagram of Experimental Burner

As shown from Figure 1 it consists of two concentric tubes and combustible gas flowing upward through the inner tube of radius L also having an opening A-A. The air flows upward again on the outer tube with the same velocity as the combustible gas due to keeping the rate of flow in the ratio of L^2 to R^2-L^2.If the burner is ignited at A-A with a steady flame of definite shape therefore flames in cylindrical tubes are called cylindrical flames. The authors replaced the cylindrical tubes with two inner walls and two outer walls which they termed flat flames and the ratio of gas to air was kept at L to L-R for both velocities to be constant. An analytical solution was then generated.

Method

S.P. Burke and T.E.W. Schumann developed analytical solutions for both the cylindrical flames and the flat flames. The paper didn't state which method they used to solve for the analytical solutions they got so I had to apply what we learned in class (ChEn 533) in order to solve for the analytical solution reproducing what they got so I ended up just focusing on the flat flames. In my approach I used S.P. Burke and T.E.W. Schumann's paper, Dr. Tree's class notes and William Deen's textbook as references.

My method starts by stating the assumptions which were basically what the authors stated and adding mine that since it's a solid therefore all v_x , v_y , v_z go to zero. I then stated the initial and boundary conditions which are the same as the ones the authors stated which included. After applying all the assumptions, the governing equation then simplified to:

$$\frac{\partial C}{\partial t} = k\left(\frac{\partial^2 C}{\partial x^2}\right) \tag{1}$$

Since v is constant, therefore v = yt and let x = r, the governing equation further simplifies to:

$$v\frac{\partial C}{\partial y} \;=\; k\frac{\partial^2 C}{\partial r^2} \tag{2}$$

I then figured that this a transient type of a problem that leads to a steady state condition thus I decided to solve for the concentration in two parts, first for the steady state term followed by the transient term then adding them together. For the steady state term, I first transformed the initial conditions and I managed to obtain θ_1 =1 and θ_2= - C_2/ iC_0 .After integrating twice and applying the transformed initial conditions which are basically constants, I obtained my steady state solution as equation (3):

$$\boldsymbol{\frac{\partial^2 \theta_{ss}}{\partial r^2} \;=\; 0 \rightarrow \theta_{ss} = \frac{L}{R} - \frac{C_2}{iC_o}} \tag{3}$$

Now for the transient part I then used the Finite Fourier Transform (FFT) method which I learned in class by first breaking my differential equation into two parts that is first the LHS and secondly the RHS and then solved each separately before combing the two. The boundary conditions used in the paper where $\frac{dC}{dr} = 0 \; when \; r = 0 \; and \; r = R$, therefore I ended up using Case 4 from table 5-2(*William Deen's textbook*) in which $\phi_n = \sqrt{\frac{2}{R}}cos\frac{n\pi r}{R} \quad n = 1,2, \dots \dots \dots \dots$

The LHS simplified easily because we could factor out the derivative and then we ended up with something that we had seen before:

$$\frac{\partial}{\partial y}\int_0^1 \theta \sqrt{\frac{2}{R}}cos\frac{n\pi r}{R}dr \rightarrow \frac{\partial C_n(y)}{\partial y} \tag{4}$$

However, we could not do the same for the RHS since we are integrating w.r.t r hence we cannot pull out of the integral any term with r.I then applied integration by parts twice inorder to simplify the RHS, from the first integration by parts I obtained:

$$\int_0^1 \frac{k}{v}\frac{\partial^2 \theta}{\partial r^2}\sqrt{\frac{2}{R}}cos\frac{n\pi r}{R}dr \rightarrow \sqrt{\frac{2}{R}}cos\frac{n\pi r}{R}\frac{\partial \theta}{\partial r}\Big|_0^1 + \int_0^1 \frac{\partial \theta}{\partial r}\sqrt{\frac{2}{R}}\frac{n\pi}{R}sin\frac{n\pi r}{R}dr \tag{5}$$

From the second integration by parts and evaluating equation (6) at r=0 and r=L, at this point I ended up getting something that I had seen before which then ended up getting rid of my integral sign therefore I obtained;

$$\int_0^1 \frac{\partial\theta}{\partial r}\sqrt{\frac{2}{R}}\frac{n\pi}{R}sin\frac{n\pi r}{R}dr \to \theta\sqrt{\frac{2}{R}}\frac{n\pi}{R}sin\frac{n\pi r}{R}\Big|_{r=0}^{r=L} - \int_0^1 \theta\sqrt{\frac{2}{R}}\frac{n^2\pi^2}{R^2}cos\frac{n\pi r}{R}dr \tag{6}$$

$$\to \frac{k}{v}\sqrt{\frac{2}{R}}\frac{n\pi}{R}sin\frac{n\pi L}{R} - \frac{k}{v}\frac{n^2\pi^2}{R^2}C_n(y)$$

After combining the RHS and LHS I then obtained a solvable PDE.

$$\frac{\partial C_n(y)}{\partial y} = \frac{k}{v}\sqrt{\frac{2}{R}}\frac{n\pi}{R}sin\frac{n\pi L}{R} - \frac{k}{v}\frac{n^2\pi^2}{R^2}C_n(y) \tag{7}$$

The general solution for Homogeneous Differential Equations with constant coefficients is $C_n(y) = Se^{rx}$ According to table B-1(A), (*William Deen's textbook, page 641*) Therefore after solving for constant S and e^{rx} we obtain:

$$S = \sqrt{\frac{2}{R}}\frac{R}{\pi n}sin\frac{\pi nL}{R} \quad , \quad e^{rx} = e^{-\frac{\pi^2 n^2 ky}{vR^2}} \to C_n(y) = \sqrt{\frac{2}{R}}\frac{R}{\pi n}sin\frac{\pi nL}{R}\sqrt{\frac{2}{R}}cos\frac{n\pi r}{R}e^{-\frac{\pi^2 n^2 ky}{vR^2}} \tag{8}$$

The solution of the transient term then simplifies to:

$$\theta_t = \sum_{n=1}^{\infty}\frac{2}{\pi n}sin\frac{\pi nL}{R}cos\frac{n\pi r}{R}e^{-\frac{\pi^2 n^2 ky}{vR^2}}$$

Recalling that $\theta(r,y) = \theta_{ss} + \theta_t$ and dimensionalizing the final solution using $\theta = \frac{C}{C_o}$, I then reproduced the same analytical solution they got using what we learnt in class as shown in equation (9).

$$C = \frac{C_o L}{R} - \frac{C_2}{i} + \frac{C_o 2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{\pi n L}{R}\cos\frac{n\pi r}{R}e^{-\frac{\pi^2 n^2 ky}{vR^2}} \tag{9}$$

As the authors did in the paper setting C =0 and r = x, I then lumped all the constants together forming a dimensionless number E, which is based on the concentrations of the combustible gas, C_2 and the ratio L/R as shown below. As the authors did , I then used the same equation to solve for E at different x and y values and ended up generating the same E-y curves they got using python at different x values as shown in Figure 2.

$$\sum_{n=1}^{\infty}\frac{1}{n}sin\frac{\pi nL}{R}cos\frac{n\pi x}{R}e^{-\frac{\pi^2 n^2 ky}{vR^2}} = \frac{\pi}{2}\left\{\frac{C_2}{iC_o} - \frac{L}{R}\right\} = E \tag{10}$$

Results and Discussion

The authors considered a particular case supposing:

R	1 inch	k	0.0763 inch^2/s	C_2	0.21	V	1.33 inch^2/s
L	1/6 inch	C_o	1	i	2		

Table 1

Now based on case above the value of E = -0.113 and if the air is enriched containing 50% oxygen is used instead, E = 0. 052 which shows that C_2 concentration is directly proportional to E. Figure 2 shows the relationship between E, y and x which I generated using python matches what they produced back in 1928.

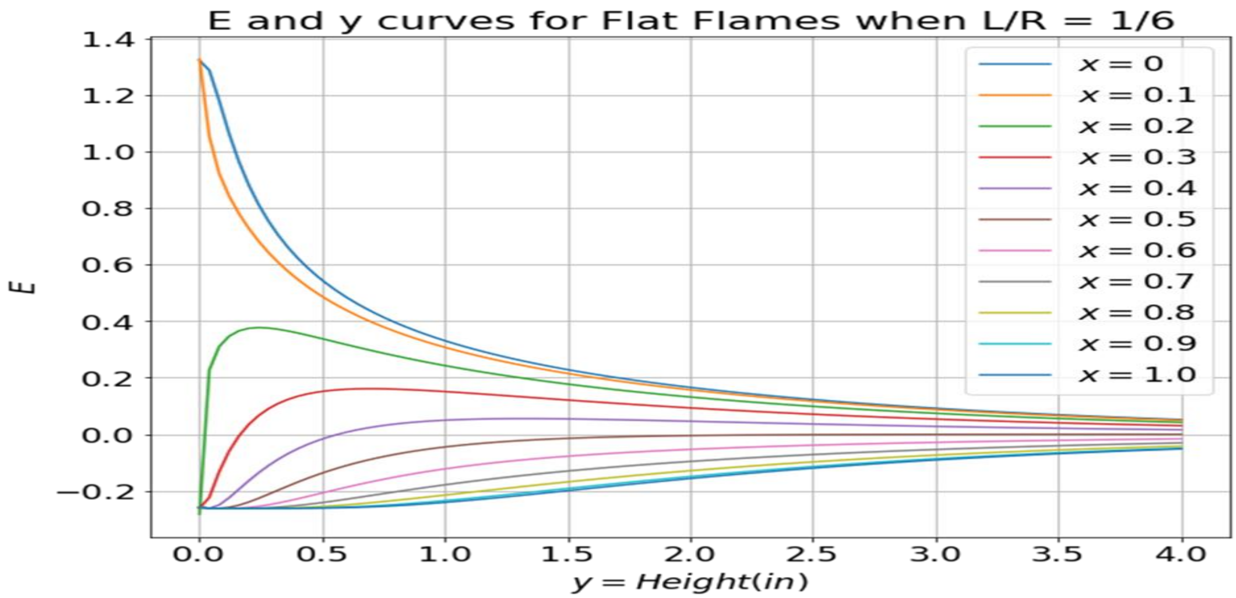


Figure 2 E-y curves for flat flames.

From Figure 2 the graph x=0 shows the height of an overventilated flame which is decreasing as E increases and the graph x=1.0 shows the height of an underventilated flame which increasing with increasing E. Given a scenario in which a portion of the combustible gas is substituted by an inert gas this will result in a decrease in C_o which causes an increase in E as shown in equation (10) ,this will result in a taller underventilated flames and shorter overventilated flames.

Conclusion

Experiments were conducted as well, and it was concluded that substitution of an inert caused an elongation of the flame in all cases. The diffusion flame mathematical solution provided the same results with the experiments hence proving that transport phenomena can be applied to real life problems. The study of diffusion flames has assisted in ensuring safety in industry such as preventing backfiring and explosions in furnaces or flares.

DIFFUSION FLAMES

APPENDIX

Assumptions:

- v_{gas} and v_{air} up the tube in the region of the flame is constant
- The coefficient of interdiffusion of the two gas streams is constant
- The interdiffusion is wholly flat
- Mixture of the two gas streams occurs by diffusion only

Definitions

- r = distance in the x-direction
- y = vertical distance above orifice of inner parallel plates
- k = coefficient of interdiffusion
- C_0 = initial concentration of combustible gas
- $C_2 = - C_2/i$ concentration of oxygen ,negative combustible gas
- i is the number of molecules of oxygen which combine with one molecule of combustible gas to effect complete combustion

Initial and Boundary Conditions

- $C = C_0$ from r=0 to r=L at y=0 I.C
- $C =-C_2$ from r=L to r=R at y=0 I.C
- $\frac{dC}{dr} = 0$ when r=0 and r=R B.C

Governing Equation: Rectangular Coordinates

Table 2-4 pg(42) : $\frac{\partial C}{\partial t} + v_x \frac{\partial C}{\partial x} + v_y \frac{\partial C}{\partial y} + v_z \frac{\partial C}{\partial z} = k\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right) + R_{vi}$

since v is constant, therefore v = yt and let x = r

$$v\frac{\partial C}{\partial y} = k\frac{\partial^2 C}{\partial r^2}$$

$$\theta = \theta_{ss} + \theta_t$$

Considering θ_{ss} part

Let r = $\frac{L}{R}$, θ = C/ C_0

Transforming BCs

$C_0 \theta_1 = C_0 \therefore \theta_1 = 1$

$C_0 \theta_2 = C_2 \therefore \theta_2 = - C_2 / iC_0$

$$\frac{\partial^2 \theta_{ss}}{\partial r^2} = 0$$

Integrating twice :

$\theta_{ss} = K_1 r + K_2$, $K_1 = 1$, $K_2 = -\frac{C_2}{iC_o}$, the BC are also constants at steady state

$$\theta_{ss} = \frac{L}{R} - \frac{C_2}{iC_o}$$

Considering ϑ_t part using the FFT Method

Table 5-2 Case 4(page 168): $\phi_n = \sqrt{\frac{2}{R}} cos \frac{n\pi r}{R}$, n = 1,2,.................

$$\theta(r, y) = \sum_{n=1}^{\infty} C_n(y) \phi_n(r) dr$$

$$\frac{\partial C}{\partial y} = \frac{k}{v} \frac{\partial^2 C}{\partial r^2}$$

$$C_n(y) = \int_0^1 \theta(r, y) \ \phi_n(r) dr$$

$$C_n(y) = \int_0^1 \theta \sqrt{\frac{2}{R}} cos \frac{n\pi r}{R} dr$$

$$\frac{\partial \theta}{\partial y} = \frac{k}{v} \frac{\partial^2 \theta}{\partial r^2}$$

LHS :

$$\int_0^1 \frac{\partial \theta}{\partial y} \sqrt{\frac{2}{R}} cos \frac{n\pi r}{R} dr$$

$$\frac{\partial}{\partial y} \int_0^1 \theta \sqrt{\frac{2}{R}} cos \frac{n\pi r}{R} dr$$

$$\frac{\partial C_n(y)}{\partial y}$$

RHS:

$$\int_0^1 \frac{k}{v}\frac{\partial^2\theta}{\partial r^2}\sqrt{\frac{2}{R}}cos\frac{n\pi r}{R}dr$$

$u = \sqrt{\frac{2}{R}}\,cos\frac{n\pi r}{R}$ $\qquad du = -\sqrt{\frac{2}{R}}\frac{n\pi}{R}sin\frac{n\pi r}{R}$

$dv = \frac{\partial^2\theta}{\partial r^2}$ $\qquad v = \frac{\partial\theta}{\partial r}$

$Applying\ Integration\ By\ Parts : \int udv = uv - \int vdu$

$$\sqrt{\frac{2}{R}}\,cos\frac{n\pi r}{R}\frac{\partial\theta}{\partial r}\Big|_0^1 + \int_0^1 \frac{\partial\theta}{\partial r}\sqrt{\frac{2}{R}}\,\frac{n\pi}{R}\,sin\frac{n\pi r}{R}dr$$

$u = sin\frac{n\pi r}{R}$ $\qquad du = \frac{n\pi}{R}\,cos\frac{n\pi r}{R}$

$dv = \frac{\partial\theta}{\partial r}$ $\qquad v = \theta$

$$\theta\sqrt{\frac{2}{R}}\,\frac{n\pi}{R}sin\frac{n\pi r}{R}\Big|_{r=0}^{r=L} - \int_0^1 \theta\sqrt{\frac{2}{R}}\,\frac{n^2\pi^2}{R^2}\,cos\frac{n\pi r}{R}dr$$

$$\frac{k}{v}\sqrt{\frac{2}{R}}\,\frac{n\pi}{R}sin\frac{n\pi L}{R} - \frac{k}{v}\,\frac{n^2\pi^2}{R^2}C_n(y)$$

Combining both LHS and RHS

$$\frac{\partial C_n(y)}{\partial y} = \frac{k}{v}\sqrt{\frac{2}{R}}\,\frac{n\pi}{R}sin\frac{n\pi L}{R} - \frac{k}{v}\,\frac{n^2\pi^2}{R^2}C_n(y)$$

General solution for Homogeneous Differential Equations solution with constant coefficients

$C_n(y) = Ce^{rx}$ According to table B-1(A) page 641

$$C = \sqrt{\frac{2}{R}}\,\frac{R}{\pi n}sin\frac{\pi nL}{R} \quad , \quad e^{rx} = e^{-\frac{\pi^2 n^2 ky}{vR^2}}$$

$$C_n(y) = \sqrt{\frac{2}{R}}\frac{R}{\pi n}sin\frac{\pi nL}{R}\sqrt{\frac{2}{R}}cos\frac{n\pi r}{R}e^{-\frac{\pi^2 n^2 ky}{vR^2}}$$

APPENDIX

$$C_n(y) = \frac{2}{\pi n} sin\frac{\pi nL}{R} cos\frac{n\pi r}{R} e^{-\frac{\pi^2 n^2 ky}{vR^2}}$$

$$\theta_t = \sum_{n=1}^{\infty} \frac{2}{\pi n} sin\frac{\pi nL}{R} cos\frac{n\pi r}{R} e^{-\frac{\pi^2 n^2 ky}{vR^2}}$$

$$\theta = \theta_{ss} + \theta_t$$

$$\theta = \frac{L}{R} - \frac{C_2}{iC_o} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} sin\frac{\pi nL}{R} cos\frac{n\pi r}{R} e^{-\frac{\pi^2 n^2 ky}{vR^2}}$$

$$C = \frac{C_o L}{R} - \frac{C_2}{i} + \frac{C_o 2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} sin\frac{\pi nL}{R} cos\frac{n\pi r}{R} e^{-\frac{\pi^2 n^2 ky}{vR^2}}$$

Putting C = O and r = x

$$E = \frac{\pi}{2}\left(\frac{C_2}{iC_o} - \frac{L}{R}\right) = \sum_{n=1}^{\infty} \frac{1}{n} sin\frac{\pi nL}{R} cos\frac{n\pi r}{R} e^{-\frac{\pi^2 n^2 ky}{vR^2}}$$

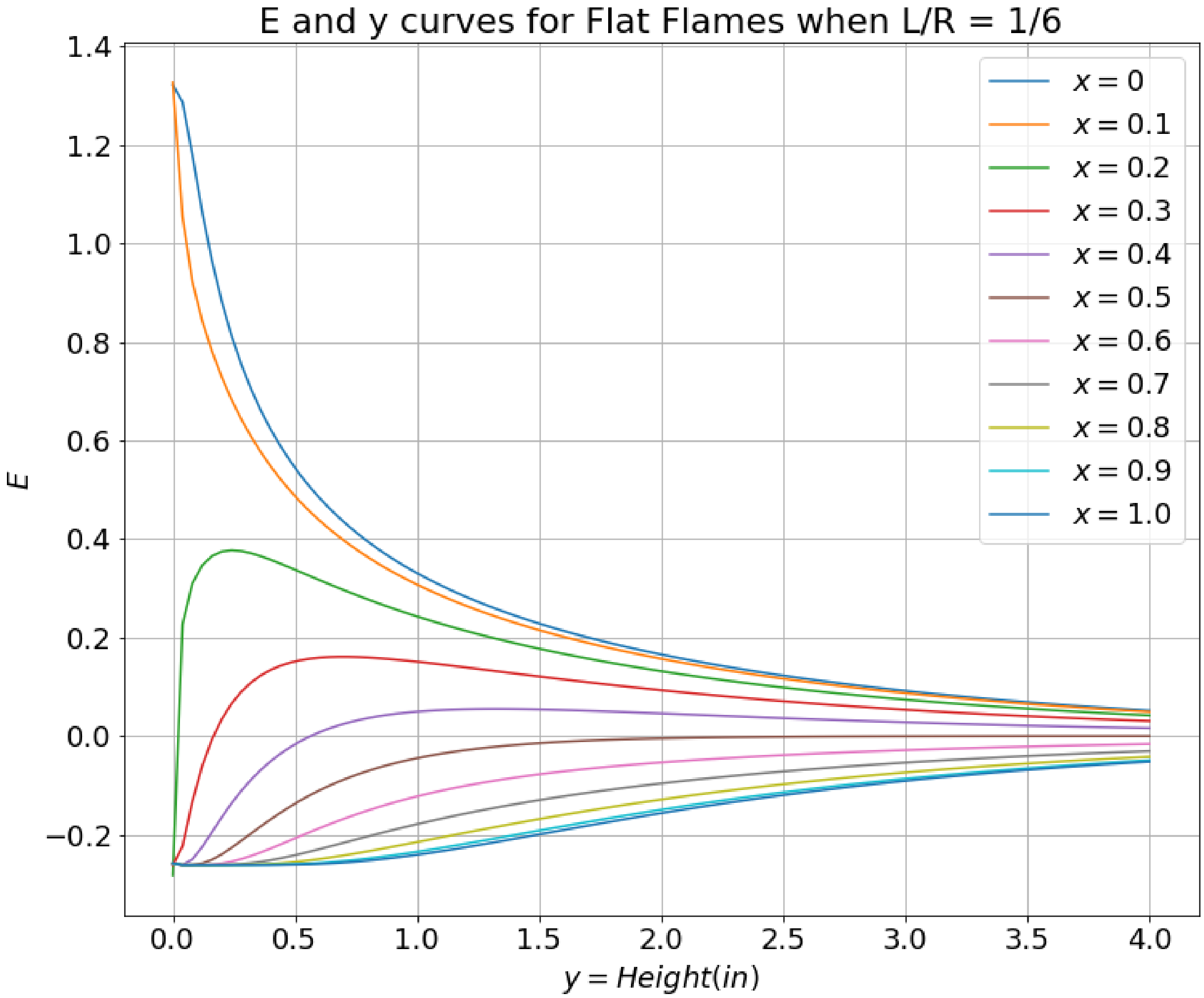
In [1]:
```python
# %load "http://che.byu.edu/imports.py"
import numpy                as np
import matplotlib.pyplot as plt
%matplotlib inline
from scipy.optimize      import fsolve, curve_fit
from scipy.integrate     import odeint, quad
from scipy.interpolate   import interp1d
from scipy.misc          import derivative
import scipy.constants   as const
import sympy             as sp
sp.init_printing()
import glob
import time,math
from scipy.stats import t
#import pint; u = pint.UnitRegistry()
```

In [7]:
```python
C1 = 1
R  = 1    #in
L  = 1/6  #in
k = 0.0763#in**2/s
C2 = 0.21
i = 2
v = 1.33  #in/s
C0 = C1+C2/i
```

In [3]:

```
Nx = 101
x = np.linspace(1,0, Nx)
y = np.linspace(0,4,Nx)
Nmax = 1001
#E1 = np.linspace(-0.3,0.4,8)
def f(E):
    f_sum = np.zeros(Nx)
    for n in range(1, Nx):
        f_sum += 1/n*np.sin(np.pi*n*L/R)*np.cos(np.pi*n*0/R)*np.exp(-k*n**2*
    return f_sum
E =fsolve(f,np.ones(len(x)))

def f1(E):
    f_sum = np.zeros(Nx)
    for n in range(1, Nx):
        f_sum += 1/n*np.sin(np.pi*n*L/R)*np.cos(np.pi*n*0.1/R)*np.exp(-k*n**
    return f_sum
E1 =fsolve(f1,np.ones(len(x)))

def f2(E):
    f_sum = np.zeros(Nx)
    for n in range(1, Nx):
        f_sum += 1/n*np.sin(np.pi*n*L/R)*np.cos(np.pi*n*0.2/R)*np.exp(-k*n**
    return f_sum
E2 =fsolve(f2,np.ones(len(x)))

def f3(E):
    f_sum = np.zeros(Nx)
    for n in range(1, Nx):
        f_sum += 1/n*np.sin(np.pi*n*L/R)*np.cos(np.pi*n*0.3/R)*np.exp(-k*n**
    return f_sum
E3 =fsolve(f3,np.ones(len(x)))

def f4(E):
    f_sum = np.zeros(Nx)
    for n in range(1, Nx):
        f_sum += 1/n*np.sin(np.pi*n*L/R)*np.cos(np.pi*n*0.4/R)*np.exp(-k*n**
    return f_sum
E4 =fsolve(f4,np.ones(len(x)))

def f5(E):
    f_sum = np.zeros(Nx)
    for n in range(1, Nx):
        f_sum += 1/n*np.sin(np.pi*n*L/R)*np.cos(np.pi*n*0.5/R)*np.exp(-k*n**
    return f_sum
E5 =fsolve(f5,np.ones(len(x)))

def f6(E):
    f_sum = np.zeros(Nx)
    for n in range(1, Nx):
        f_sum += 1/n*np.sin(np.pi*n*L/R)*np.cos(np.pi*n*0.6/R)*np.exp(-k*n**
    return f_sum
E6 =fsolve(f6,np.ones(len(x)))

def f7(E):
    f_sum = np.zeros(Nx)
```

```
    for n in range(1, Nx):
        f_sum += 1/n*np.sin(np.pi*n*L/R)*np.cos(np.pi*n*0.7/R)*np.exp(-k*n**
    return f_sum
E7 =fsolve(f7,np.ones(len(x)))

def f8(E):
    f_sum = np.zeros(Nx)
    for n in range(1, Nx):
        f_sum += 1/n*np.sin(np.pi*n*L/R)*np.cos(np.pi*n*0.8/R)*np.exp(-k*n**
    return f_sum
E8 =fsolve(f8,np.ones(len(x)))

def f9(E):
    f_sum = np.zeros(Nx)
    for n in range(1, Nx):
        f_sum += 1/n*np.sin(np.pi*n*L/R)*np.cos(np.pi*n*0.9/R)*np.exp(-k*n**
    return f_sum
E9 =fsolve(f9,np.ones(len(x)))

def f10(E):
    f_sum = np.zeros(Nx)
    for n in range(1, Nx):
        f_sum += 1/n*np.sin(np.pi*n*L/R)*np.cos(np.pi*n*1.0/R)*np.exp(-k*n**
    return f_sum
E10 =fsolve(f10,np.ones(len(x)))

plt.rc('font',size=18)
plt.figure(figsize=(12,10))


plt.plot(y,E*100,label='$ x=0$')
plt.plot(y,E1*100,label='$ x=0.1$')
plt.plot(y,E2*100,label='$ x=0.2$')
plt.plot(y,E3*100,label='$ x=0.3$')
plt.plot(y,E4*100,label='$ x=0.4$')
plt.plot(y,E5*100,label='$ x=0.5$')
plt.plot(y,E6*100,label='$ x=0.6$')
plt.plot(y,E7*100,label='$ x=0.7$')
plt.plot(y,E8*100,label='$ x=0.8$')
plt.plot(y,E9*100,label='$ x=0.9$')
plt.plot(y,E10*100,label='$ x=1.0$')
plt.legend()
plt.grid()
plt.title('E and y curves for Flat Flames when L/R = 1/6')
plt.xlabel('$y=Height (in)$')
plt.ylabel('$E$')
plt.show()
plt.tight_layout()
```



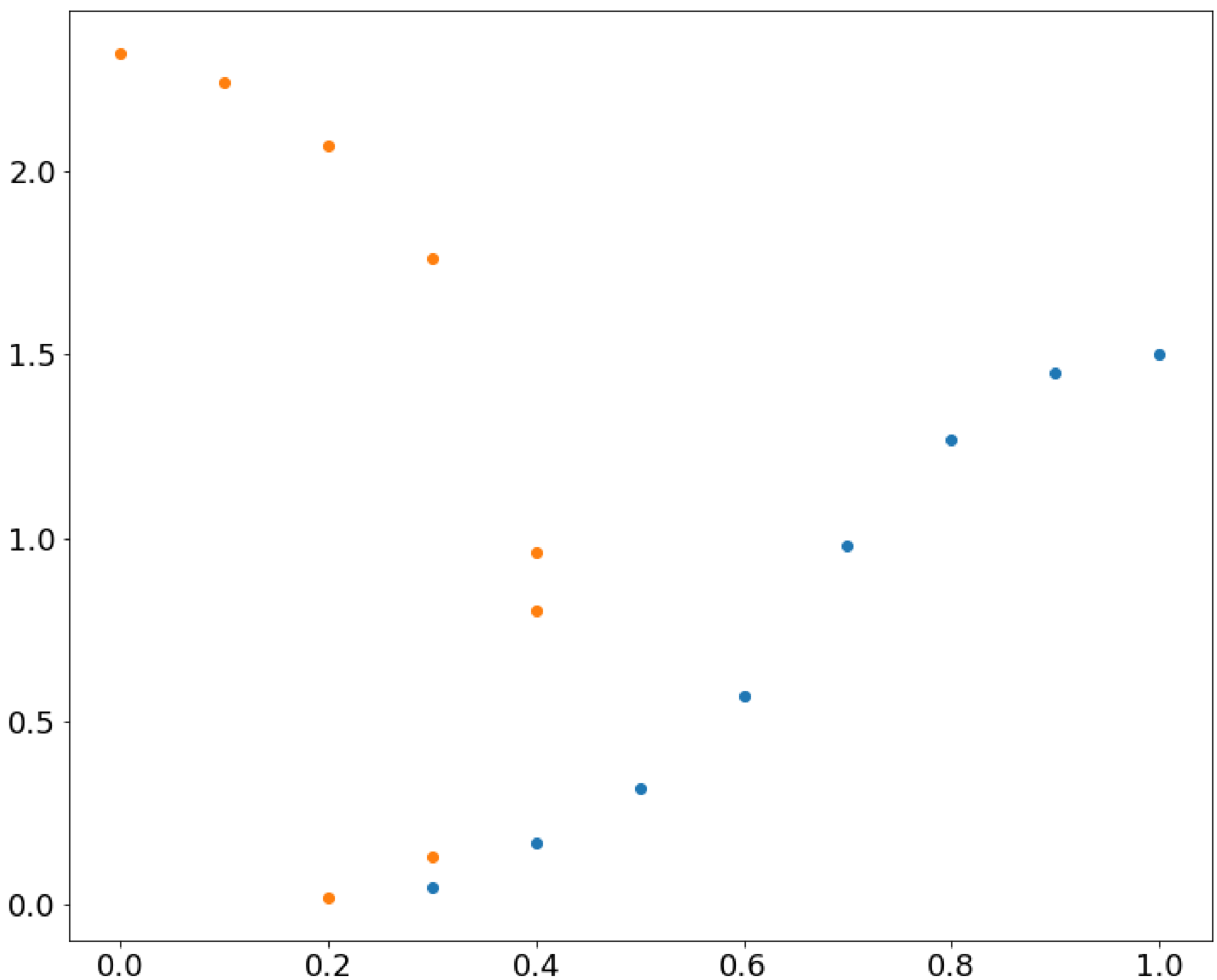
<Figure size 432x288 with 0 Axes>

In [10]:

```
#E = -0.113

x1 = np.array([1,0.9,0.8,0.7,0.6,0.5,0.4,0.3])
y1 = np.array([1.50,1.45,1.27,0.98,0.57,0.32,0.17,0.05])
#When E = 0.052
x2 = np.array([0,0.1,0.2,0.2,0.3,0.3,0.4,0.4])
y2 = np.array([2.32,2.24,2.07,0.02,1.76,0.13,0.96,0.80])
plt.figure(figsize=(12,10))
plt.plot(x1,y1,'o')
plt.plot(x2,y2,'o')
```

Out[10]:

```
[<matplotlib.lines.Line2D at 0x217d767d6a0>]
```



In [ ]:

In [ ]: