III.

THE STABILITY OR INSTABILITY OF THE STEADY MOTIONS OF A PERFECT LIQUID AND OF A VISCOUS LIQUID. Part II.: A VISCOUS LIQUID.

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Introduction and Summary of Contents.

In Part I.* reference was made to a well-known difficulty in reconciling theory and experiment in the case of the steady motion of liquids. The flow through pipes and between concentric cylinders, one of which is rotated, had been found experimentally to be unstable if the velocity is great enough; while, on the other hand, Lord Rayleigh had shown that, in these cases, if the effect of viscosity be neglected in the disturbed motion, the fundamental free disturbances are strictly periodic, the values of the "free periods" being real. An explanation of the difficulty was given by showing that it is necessary to push Lord Rayleigh's investigations a step farther by resolving a disturbance into its constituent fundamental ones by *quasi*-Fourier analysis, and that, when this is done for disturbances of initially simple type in some of the most important and simplest cases of flow, it is found that the disturbance will, for suitable values of the constants, increase very much, so that the motion is practically unstable.

The present investigation attempts to discover how far this conclusion must be modified when viscosity is taken account of.

It may be stated at once that I have not succeeded in throwing much additional light on this matter; but a good deal of the work had been done before I discovered that the slight extension of Lord Rayleigh's analysis which is contained in Part I. would explain the difficulty, at least qualitatively;† and I therefore decided to carry the investigation as far as I could: I may moreover plead that I found some portions of the analysis interesting on their own account.

* Proc. R.I.A., vol. xxvii., Section A, No. 2.

† I consider that a proof of instability for a perfect liquid is a proof of instability also for a viscous liquid *if the viscosity be small enough.*

Chapter I., pp. 80–94, deals with Lord Kelvin's investigations.*

The two problems which he discussed having been described in Art. 1, p. 80, an abstract is given in Art. 2, pp. 80–83, of one of his proofs that an infinitely wide stream of finite depth and uniform vorticity is stable; this solution, following Lord Rayleigh, I describe as a "special" solution in contradistinction to another which he indicated in a subsequent paper. As far at least as the velocity-component in the direction of the depth is concerned, Lord Kelvin first obtains a solution, (v), of the differential equation which satisfies the most general initial conditions throughout, but violates the permanent boundary-conditions at the top and bottom of the stream; he then adds to this solution a "forced" disturbance, ($\mathfrak{v}$), which would be caused throughout the stream by exactly reversing this outstanding boundary disturbance, and, by addition, thus obtains a solution which does satisfy the boundary-conditions. The "forced" disturbance is obtainable as an integration of an infinity of constituents each of which is simply-periodic in the time, and the constituents are to be chosen by a Fourier analysis, valid between the times $t = -\infty$ and $t = +\infty$ so as to satisfy the boundary-conditions $\mathfrak{v} = 0$ from $t = -\infty$ till $t = 0$, and $\mathfrak{v} = -\mathrm{v}$ from $t = 0$ till $t = \infty$. The v solution is composed of one or more terms, each of which has a factor which involves the time exponentially, the index being essentially negative, and eventually varying as the cube of the time; thus v diminishes indefinitely; and Lord Kelvin states that hence the "forced" disturbance $\mathfrak{v}$, which rises gradually from zero at $t = 0$, also diminishes indefinitely, and concludes that the steady motion is stable.

Art. 3, p. 83, contains a brief account of another proof of stability in the same motion, which Lord Kelvin indicates in his discussion of the second of the two problems which he discussed.

Art. 4, p. 84, gives Lord Rayleigh's adverse criticism of the second solution, in which he points out that Lord Kelvin has merely shown the possibility of obtaining forced vibrations of arbitrary (real) frequency, and that this constitutes no proof of stability, it being possible to do this in the case of a pendulum displaced from a position of unstable equilibrium.

Art. 5, pp. 84–85, gives remarks by Lord Rayleigh on the "special" solution in which he appears to accept it.

In Art. 6, p. 85, it is pointed out, however, that the "special" solution involves a tacit assumption that the "forced" disturbance, $\mathfrak{v}$, vanishes everywhere throughout the liquid at the time $t = 0$.

In Art. 7, p. 86, it is argued that this assumption is legitimate if

* Phil. Mag., August and September, 1887.

is known that the fundamental free disturbances have stability of the common exponential type, but that it would not be true if the contrary were the case; and in Art. 8, pp. 86–88, a simple instance is taken of a system having only one coordinate in which this argument is seen to be correct.

In Art. 9, p. 88, it is pointed out besides, that, except at the boundaries, it is not known that the "forced" disturbance, $\mathfrak{v}$, *does* diminish indefinitely.

It is accordingly held that Lord Kelvin has not proved stability, even for infinitesimal disturbances.

As the fundamental modes of disturbance do, as is shown in Chapter II., possess stability of the simple exponential character, the "special" solution is, I believe, as a matter of fact, the solution for a given initial disturbance; if this be a simple trigonometrical function of the coordinates, the form of v is simple; but that of the "forced" disturbance, $\mathfrak{v}$, in no case appears capable of being readily calculated. It is urged, however, in Art. 10, pp. 88–90, that this solution actually proves that for sufficiently small viscosity or sufficiently great velocity the motion is unstable; for under such circumstances v, considered alone, will increase very much if the constants are properly chosen, the possible ratio being limited only by friction; and it is held that the fact that v violates the boundary-conditions is of little importance if the wave-lengths in all directions are sufficiently small. The boundary-conditions being that the velocity perpendicular to the depth of the stream and its gradient in the same direction should vanish, it is seen moreover that it is quite easy to add to v a term which gives a solution satisfying either one of these conditions or the other, but not both. (If the former be chosen, the solution thus obtained includes as a limiting case that given in Part I. for the same problem in the absence of viscosity.)

In Art. 11, pp. 90–92, numerical values corresponding to the circumstances under which instability has been actually observed to set in under somewhat similar circumstances are substituted in the two-dimensioned form of the first of these two modifications of the "special" solution; it appears that it would not be possible for the kinetic energy of the relative motion of any disturbance of the simple type in question to increase to more than about four times its original value.

And in Art. 12, p. 93, the same is done for the second modification; and it is seen that an initial disturbance of the same type, but with different constants, might increase about ten-thousand-fold.

In Chapter II., pp. 95–121, the fundamental free disturbances of this same steady motion are discussed.

The preliminary analysis is, of course, substantially that given by Lord Kelvin in the "special" solution: supposing the plane boundaries to be

$y = \pm a$, and the steady velocity to be βy in the x-direction, the y-velocity in the disturbed motion is taken to be $v = Ve^{pt+i(lx+nz)}$, where l and n are arbitrarily assigned and p is to be found. The differential equation shows that $\nabla^2 v$ is of the form:—

$$u^{\frac{1}{3}}\{AJ_{\frac{1}{3}}(u) + BJ_{-\frac{1}{3}}(u)\} \quad \text{where } u \text{ is of the form} \quad (Cy + C')^{\frac{3}{2}};$$

if the boundary-conditions should include the vanishing of $\nabla^2 v$, it is thus seen that the investigation is very much simpler than for the natural conditions $v = 0$, $dv/dy = 0$; and accordingly this case is discussed in detail.

In Art. 13, p. 95, the equation giving the values of p (the period-equation) is derived.

In Art. 14, p. 96, in view of a remark of Lord Rayleigh's which appears to suggest that it may not be possible to obtain disturbances which do vary as e^{pt}, it is first proved, or rather rendered probable—for the demonstration is not rigorous—that this equation has an infinite number of roots; this follows by making use of the approximate forms of the Bessel functions for large values of the variable.

In Art. 15, p. 99, it is proved directly from the differential equation that all possible values of p must have a real negative part, and that the imaginary part lies between the extreme limits found when there is no viscosity.

Art. 16, p. 100, gives a rigorous proof that for all values of l, n, there are an infinite number of real values of p.

Art. 17, p. 101, indicates briefly a proof that if la is small enough, all the values of p are real, and given approximately by a comparatively simple algebraic equation; this proof is developed rigorously in Art. 18, p. 102, which contains as a necessary step an investigation of the number of roots inside a circular contour of large radius having the origin as centre, this investigation and its result holding good, whatever the value of la.

In Art. 19, p. 106, the double roots are considered; it is shown that a double root occurs when, and only when, a certain multiple of $(l\beta a^3/\nu)^{\frac{1}{3}}$ is a root of $J_{\frac{1}{3}}(u) = 0$, ν denoting the kinematic viscosity; and, in Art. 20, p. 108, it is proved that, as l increases through such a value, two real roots do actually disappear; while in Art. 21, p. 111, approximate expressions are obtained for the complex roots. It is seen that all the roots, real and complex, are accounted for. There are thus a definite finite number of complex roots, and for them the values of $p + \nu(l^2 + n^2)$ lie close to two straight lines which contain an angle of $2\pi/3$. When the disturbance is oscillatory, its time is independent of n.

In Art. 22, p. 111, it is proved that, in the most persistent disturbance, v is a function of y only; i.e., l and n are zero.

Art. 23, p. 113, contains two fundamental equations showing how to discover the coefficients of the *quasi*-Fourier expansion of an arbitrary function of y in a series consisting of the infinity of V's which correspond to given values of l, n; it seems reasonable to assume the possibility of such an expansion; I am quite unable to prove it. I have failed in the endeavour to apply this analysis quantitatively to the case of a disturbance of simple type, as was done in Part I., Chap. I., Arts. 4–8.

In Art. 24, p. 115, a brief reference is made to the case in which the boundary-conditions $\nabla^2 v = 0$ are replaced by $d/dy \,.\, \nabla^2 v = 0$.

The much more difficult case in which the boundary-conditions are

$$v = 0, \quad dv/dy = 0$$

is taken up in Art. 25, p. 117; it is proved that the imaginary part of p lies between the same limits as before. I have failed, however, to obtain any direct proof from the differential equation itself that p has a negative real part, and also to obtain any equations by the aid of which the Fourier analysis of an arbitrary disturbance can be performed. There is frequently a connexion between these two questions; a fundamental equation of Bessel-Fourier analysis,* for instance, serves equally to prove that all zeroes of the Bessel function of order greater than -1 are real; and, though equation (63) of Art. 23 does not show the roots to have a real negative part with the boundary-conditions $\nabla^2 v = 0$, the two results have been obtained by similar methods. Probably some simple proof that p has a negative real part in the present case will be discovered; but it seems possible that no simple theorem relating to Fourier expansion may hold. Similar difficulties may arise to a certain extent, even for a system having only a finite number of coordinates; in some such cases the proof of stability for fundamental disturbances is much more difficult than that of the reality of the roots of the determinantal equation which is met in the corresponding problem of displacement from equilibrium, and the period equation may have to be examined as carefully as any other algebraic equation, the fact that it arises in a dynamical problem being regarded as a mere accident; also, when, in steady motion, the fundamental determinant is unsymmetrical, and there exist forces of resistance proportional to the velocities, no rule appears to be known for abbreviating the labour of solving the simultaneous simple equations which determine the coefficients of the fundamental disturbances making up a given initial one.

* I. e. the equation

$$\int_0^a J_n(\kappa r)\, J_n(\lambda r)\, r dr = 0,$$

where κa, λa are different zeroes of $J_n(x)$, and $n + 1$ is positive.

In Art. 26, the period equation is expressed in terms of integrals which involve $\nabla^2 v$, a function whose form has been already found. On the supposition that the approximate forms of the Bessel functions, for large values, may be used in this case also, I have given an approximate form of the equation appropriate to the region in which the roots actually lie. In this portion of the investigation somewhat intricate questions arose from the fact that the approximations assume different forms in different regions. Fortunately, in the region in which the roots actually occur, the difficulty is not met with in its entirety. As I am quite unable to solve this equation in the most general case, it seems undesirable to give this portion of the investigation, which is somewhat long, in full.

In Art. 27, p. 119, some results are stated. It appears that for none of the roots can the disturbance be unstable, but owing to the way in which approximations have been used, the proof indicated is not rigorous. The result of an investigation of the number of roots inside a circle of large radius round the origin is stated. The period-equation for a liquid at rest, a problem discussed by Lord Rayleigh, is obtained as a special case. A reference is made to the case in which $a(l^2 + n^2)^{\frac{1}{2}}$ is large; for the smaller values of p the roots are very nearly the same as with the boundary conditions $\nabla^2 v = 0$. Some reference is made to the general case; for such of the real roots as are remote from the complex ones, an equation is given, which, if the values of the constants were given, could be readily solved; for the others, especially the complex ones, the form is very complicated. In all cases, however, there are an infinity of real roots, and a finite, but undetermined number, which may be zero, of complex; and, roughly speaking, for these the values of $p + \nu(l^2 + n^2)$ lie in the neighbourhood of the same two lines as with the boundary-conditions $\nabla^2 v = 0$. An approximate form of the period-equation is given suitable to the case in which $a(l^2 + n^2)^{\frac{1}{2}}$ is indefinitely small, the form of the period-equation previously taken now becoming an identity; the equation giving the complex roots is still complicated.

It will be seen that, except in the case of very slow motion and in that of large values of $a(l^2 + n^2)^{\frac{1}{2}}$, the discussion is very incomplete and unsatisfactory when the boundary-conditions are that v and dv/dy should vanish.

Owing to the failure to use Fourier analysis in the simplest case,* the whole investigation elucidates the question of stability but little; for it seems unjustifiable as a mathematical proposition to infer that the steady motion of

* I. e. that in which the boundary-conditions include the vanishing of $\nabla^2 v$.

a system possessing an infinite number of coordinates is stable for an arbitrary disturbance, however small, from the stability, even when of an exponential character, of the fundamental ones into which it can be resolved; an infinite series of the type

$$\Sigma e^{-p_r t}(C_r \cos \omega_r t + S_r \sin \omega_r t),$$

like one in which no exponential factor occurs, may at some times have a value which is exceedingly great compared with its initial one, and may even become infinite. To discover how far the motion is stable for any particular disturbance, it may be necessary to obtain completely the corresponding solution, whether by Fourier analysis or otherwise. Possibly, it rarely happens that stability for the fundamental disturbances is associated with instability for those of a more general type: but this is the case in the problem under discussion, as far at least as practical stability is concerned;* this is sufficiently evident from the results of Part I., and Chap. I., Arts. 11, 12, below. It would seem improbable that any sharp criterion for stability of fluid motion will ever be arrived at mathematically. Indeed, in simpler cases of steady motion where there are only a few coordinates, although such a criterion has been laid down, it has been shown that it cannot always be relied on. It has been proved by Klein† and by Bromwich‡ that where there is exponential instability, but only slight, there may be practical stability, and *vice versa.* There is, however, this difference between such cases and the present one, that in them recourse has to be had to the terms of the second order, while here the motion is unstable, if terms of the first order only are taken into account.

Chapter III., pp. 122–138, consists of some applications of the method of Osborne Reynolds.

The method is explained in Art. 28, p. 122. Taking an arbitrary disturbance, Reynolds§ found an expression for the rate of increase of the kinetic energy of the relative motion; this is made up of two terms, of which one is essentially negative, and is the dissipation function for the relative motion; the other may be positive or negative. On equating the sum to zero, a value of the coefficient of viscosity, μ, is obtained for which the disturbance would be stationary for an instant; if the disturbance is chosen so as to make this μ as great as possible, then for any greater μ every initial disturbance must decrease; there is thus obtained an inferior limit to that value of μ which would permit

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* That is, if the viscosity is small enough.

† "The Mathematical Theory of the Top" (Princeton Lectures, 1896).

‡ "Note on Stability of Motion with an Application to Hydrodynamics," Proc. Lond. Math. Soc., xxxiii., Feb. 1901.

§ "On the Dynamical Theory of Incompressible Viscous Fluids, and the Determination of the Criterion," Phil. Trans. A, 186, Part I., 1895; Scientific Papers, ii.

a given motion to be unstable. Previous investigators by this method have selected the *type* of disturbance to some extent arbitrarily.

In Art. 29, p. 124, however, the method of variation is used to assist in discovering the proper type; it is shown that when the value of μ is the greatest for which it is possible that a disturbance should remain stationary, the velocity components in the disturbance satisfy certain differential equations.

These are applied in Art. 30, p. 124, to the uniformly-shearing stream for a two-dimensioned disturbance, supposed of definite but undetermined wave-length in the direction of flow. The differential equation to be solved in all such cases is linear and of the fourth order; in this particular instance it has constant coefficients. The boundary-conditions lead to equations determining μ; as in the other cases to be discussed, μ, so determined, has an infinite number of values; the greatest of these is taken; finally, the wave-length in the direction of flow is so chosen that this value shall be the greatest possible. The final result is $B\rho D^2/\mu = 177$, where ρ is the density, D the distance between the planes, and the steady velocity is $U = By$. H. A. Lorentz, who discussed a species of elliptic whirls, obtained the number 288 instead.*

Two cases of other boundary-conditions are discussed in Art. 31, p. 129.

Art. 32, p. 130, takes up the case of a stream flowing between *fixed* parallel planes, the second of the two problems discussed in such a different manner by Lord Kelvin, and the numerical investigations by Reynolds himself and by Sharpe are briefly described.

In Art. 33, p. 131, the more general plan which I have indicated of using Reynolds' method is applied to this case, again in two dimensions. When the velocity perpendicular to the boundaries is expanded in powers of the distance from the central plane, the differential equation gives a linear relation among the coefficients of three successive terms; there are two independent solutions in series containing only odd powers, and two in series containing only even; reasons are given justifying the choice of the latter (I confess I shrank from the labour of the double investigation). The equation which determines μ when developed from the boundary-conditions is easily solved with sufficient accuracy. Choosing the wave-length in the direction of flow so as to make this value of μ as great as possible, there results the criterion $D\overline{U}\rho/\mu = 117$, $\overline{U}$ being the mean velocity. Reynolds obtained the number 517, Sharpe 167.

Art. 34, p. 134, goes on to the case of a circular pipe, and refers to Sharpe's investigation.

* See p. 124.

And in Art. 35, pp. 135–138, the more general method is applied to a symmetrical disturbance. The differential equation is of a similar type to that in the preceding case, and is solved in a similar manner; the final result is $D\bar{U}\rho/\mu = 180$, D being the diameter of the pipe; the number obtained by Sharpe is 470. The law of velocity in this instance being $U = C'(a^2 - r^2)$, and that in the last $U = C(a^2 - y^2)$, the value I have found for C' is almost double that for C.

It is claimed that in each case the numbers I have found are true least values (but with some reservation as to the effect of end-conditions); that below them every disturbance must automatically decrease, and that above them it is possible to prescribe a disturbance which will increase for a time.

The numbers obtained above give velocities very much below those at which observers have found motions actually to become unstable; this is to be expected.

Although I cannot profess to have examined the records of the experiments carefully, it seems that the results of Reynolds[1] and of Couette[2] are to some extent contradicted by Mallock's.[3] The general result of each is that, up to a certain velocity, the motion is certainly stable, and the frictional resistance varies as the velocity: beyond this comes a region in which the motion appears at times to be stable, and at times to be unstable, the average resistance on the whole now increasing more rapidly than the first power of the velocity: if the velocity is still further increased, the motion is permanently eddying and turbulent, and the resistance is, approximately at least, proportional to the square of the velocity. Reynolds found, from experiments made on pipes of different diameters, and in which the viscosity was varied by varying the temperature, that the motion was certainly stable until $D\bar{U}\rho/\mu = 1900$. Couette gives results of experiments[4] on eight pipes of different diameters, the temperature being approximately constant. The mean value of $D\bar{U}$ is very nearly 25·4 in C. G. S. units, the range being from 22 to 28; taking μ/ρ at 13°·8 C. (the mean temperature) to be ·0118, this gives $D\bar{U}\rho/\mu = 2150$. Moreover, some of Reynolds' experiments were made with colour-bands—a method which might be expected to reveal eddies which might otherwise escape detection, and thus to give a lower limit for $\bar{U}$.

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[1] "An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and of the law of resistance in parallel channels," Phil. Trans. 1883; Scientific Papers, ii.

[2] "Etudes sur le frottement des liquides," Annales de Chimie et de Physique, 6e Série xxi., 1890.

[3] "Experiments on Fluid Viscosity," Phil. Trans., 187, 1896.

[4] L.c., p. 488.

Couette found that when a cylinder of radius 14·6395 cm. was rotated in water at 16·7° C. outside a concentric one of radius 14·3930 cm., the motion ceased to be thoroughly stable when the speed exceeded about 56 revolutions per minute; taking μ to be ·011, this corresponds to a value of $B\rho D^2/\mu = 1940$ for liquid shearing at the same rate as that in contact with the fixed cylinder. In Mallock's experiments, when a cylinder of radius 9·943 cm. was rotated outside one of 7·632 cm., it appears from a diagram that, at the temperature 0° C., the motion was not thoroughly stable when the speed exceeded about 75 revolutions per minute; this corresponds to a value of $B\rho D^2 = 204$, or, taking μ = ·018, $B\rho D^2/\mu \doteqdot 11300$. When another outer cylinder of 8·687 cm. radius was substituted, the corresponding number of revolutions was about 78, giving $B\rho D^2/\mu \doteqdot 4500$. (Up to these speeds the resistance varied as the velocity.) Moreover, Mallock states that the critical velocity he found at different temperatures was not proportional to the viscosity. "At a temperature of 50° C. the viscosity of water is only about a third of what it is at 0° C., but, at the former temperature, instability begins at a speed only of 11 or 12 per cent. less than at the latter." (His diagrams seem to indicate 15 to 20 per cent. less.)

In the experiments with different cylinders, the conditions of dynamical similarity are not satisfied; but they would appear to be practically satisfied with the same cylinders at different temperatures; (apparently conditions concerning pressure and gravity may be disregarded). Unless Mallock's results are rejected altogether, Reynolds' conclusion that in similar systems eddies appear when $\overline{U}L\rho/\mu$ exceeds some definite limit depending on the form of the apparatus (L denoting the linear dimensions), would seem to be open to doubt, despite the strong confirmation it receives from Couette's experiments.

Mallock attempted experiments in which the outer cylinder was fixed and the inner one rotated, and states that, in these circumstances, the motion seemed essentially unstable at all speeds. I have great difficulty in accepting this conclusion; and apparently the fact may just as well have been that it was found impossible to *establish* the steady motion starting from rest.

It seems remarkable too that the values of the coefficient of viscosity which Mallock deduced from his two sets of experiments differ from one another, and exceed the usually accepted values, one set being, throughout the whole range of temperatures, not much less than twice that given by Poiseuille.

In earlier experiments of a similar type by Mallock,[1] it was found that at

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[1] "Determination of the Viscosity of Water," Proc. Roy. Soc. xlv., 1888, p. 126.

all speeds the resistance could be represented as the sum of two terms, one varying as the velocity and the other as its square; the latter was attributed to the action of the ends of the rotating cylinder, and was found to become smaller and smaller as the ratio of the length to the width of the annulus increased.

[I take this opportunity of making a few corrections in Part I.:—

p. 15, l. 8, for "m" read "m^2".

p. 15, l. 3 from foot, for "a" read "*any*".

p. 25, l. 25, for "B" read "β".

p. 31, l. 19, for "$\doteqdot$" read "$=$".

p. 35, l. 17, for "ξ" read "$\dot{\xi}$".

p. 35, last line, for "$(\sqrt{5}-1)2$" read "$(\sqrt{5}-1)/2$".

p. 40, I would withdraw the opinion expressed in the final sentence which begins on this page.

p. 42, l. 21. In keeping with the last change, I would insert "lb and" before "mb".

p. 47, *et seq.* Just as the analysis of Art. 21 is simpler than that of Art. 20, so, in the disturbance discussed in Art. 18, the investigation is simpler when ka is very small, the other extreme case from that chosen.

The following electric analogy may illustrate instability of fluid motion:— In two dimensions vorticity represents electric density—stream-function, potential. Take a shearing stream with embedded positive and negative electric charges, arranged, as an extreme and simple case, like rectangles on a chess-board, the sides parallel to the direction of the stream being much longer than those across it, and the bounding-planes being kept at zero potential. Let the charges, like the vorticity, flow with the stream. When sheared so that original diagonals run right across the stream, the potential at most points towards mid-stream is much greater than originally, owing to the altered distribution of the charges.]

CHAPTER I.

LORD KELVIN'S INVESTIGATIONS, ESPECIALLY THE CASE OF A STREAM WHICH IS SHEARING UNIFORMLY.

ART. 1. *The Problems which Lord Kelvin discussed.*

THE stability or instability of steady laminar motion, when viscosity is taken into account in the disturbed motion, has been discussed by Lord Kelvin for two cases. One of these is that of a fluid undergoing simple shear, the problem which, when viscosity is ignored, formed the chief subject of Part I., Chap. I., of the present paper;* in the other, the steady velocity is a quadratic function of the distance from a plane boundary, as with a viscous fluid which is moved between two fixed parallel infinite planes by gravity or by applied pressure.

As somewhat subtle controversial matters are to be touched on in what follows, it appears desirable, with a view to facilitate the reader's comprehension of the points at issue, to give to some extent an outline of the substance of his investigation.

Lord Kelvin, in one paper,† discussed the former of the two problems alluded to; in another,‡ he attacked the latter problem on somewhat different lines, and in a foot-note indicated that this method applies equally to the former, and thus constitutes a second solution of it. It will be convenient to allude to the former solution as his "special" solution.

ART. 2. *Abstract of his Special Solution in the case of the Stream shearing uniformly.*

Referring, then, to his first paper, if we denote the plane boundaries by $y = 0$, $y = b$, suppose that the former is reduced to rest, that the velocity in the steady motion is $U = \beta y$, and that in the disturbed $U + u, v, w$, and

* Proc. R.I.A., xxvii., A. No. 2, p. 9.

† "Stability of Fluid Motion—Rectilinear Motion of Viscous Fluid between two Parallel Planes," Phil. Mag., Aug. 1887.

‡ "Stability of Motion—Broad River flowing down an Inclined Plane Bed," Phil. Mag., Sept., 1887.

denote the kinematic viscosity, or quotient of viscosity by density, by ν, the fundamental equations are

$$\left.\begin{aligned} du/dt + \beta y du/dx + \beta v &= \nu\nabla^2 u - \rho^{-1}dp/dx, \\ dv/dt + \beta y dv/dx &= \nu\nabla^2 v - \rho^{-1}dp/dy, \\ dw/dt + \beta y dw/dx &= \nu\nabla^2 w - \rho^{-1}dp/dz \\ du/dx + dv/dy + dw/dz &= 0 \end{aligned}\right\}; \tag{1}$$

and from these we obtain, by elimination,

$$(d/dt + \beta y d/dx - \nu\nabla^2)\,\sigma = 0, \tag{2}$$

where

$$\sigma = \nabla^2 v. \tag{3}$$

Ignoring, for the sake of brevity, any further reference to u, w, it is desired to obtain an expression for v, satisfying (2) and also the following initial and boundary conditions:—

when $t = 0$, v to be a given arbitrary function of x, y, z; (4)

when $y = 0$, and when $y = b$, for all values of x, z, t, both v and dv/dy to vanish. (5)

Lord Kelvin first proceeds to find a particular solution, v, of (2) which satisfies the initial conditions (4) irrespectively of the boundary conditions (5), except as follows:—

$$\mathrm{v} = 0 \text{ when } t = 0, \text{ and } y = 0 \text{ or } y = b. \tag{6}$$

He next finds another particular solution, $\mathfrak{v}$, satisfying the following initial and boundary conditions:—

$$\mathfrak{v} = 0, \quad d\mathfrak{v}/dy = 0, \quad \text{when} \quad t = 0, \tag{7}$$

$$\mathfrak{v} = -\mathrm{v}, \quad d\mathfrak{v}/dy = -\,d\mathrm{v}/dy, \quad \text{when} \quad y = 0,\ y = b. \tag{8}$$

The required complete solution will then be

$$v = \mathrm{v} + \mathfrak{v}. \tag{9}$$

To find v, Lord Kelvin remarks, that if ν were zero, the complete integral of (2) would be

$$\sigma = f(x - \beta yt, y, z), \tag{10}$$

where f is a perfectly arbitrary function, and takes therefore as a trial for a type of solution with ν not zero,

$$\sigma = Te^{i(lx + (m - l\beta t)y + nz)}, \tag{11}$$

where T is a function of t. Substituting in (2), one obtains

$$T = Ce^{-\nu t[l^2 + m^2 + n^2 - lm\beta t + l^2\beta^2 t^2/3]}, \tag{12}$$

and hence, from (3),

$$\mathrm{v} = -\frac{Te^{i(lx + (m - l\beta t)y + nz)}}{l^2 + (m - l\beta t)^2 + n^2}. \tag{13}$$

Realizing by adding solutions of this type for $\pm i$ and $\pm m$ with proper values of C, one obtains types of complete real solution

$$\text{v} = \tfrac{1}{2}k \left\{ \frac{Exp.\{-\nu t(l^2 + m^2 + n^2 - lm\beta t + l^2\beta^2 t^2/3)\}}{l^2 + (m - l\beta t)^2 + n^2} \begin{matrix}\cos \\ \sin\end{matrix} [lx + (m - l\beta t)y + nz] \right.$$
$$\left. - \frac{Exp.\{-\nu t(l^2 + m^2 + n^2 + lm\beta t + l^2\beta^2 t^2/3\}}{l^2 + (m + l\beta t)^2 + n^2} \begin{matrix}\cos \\ \sin\end{matrix} [lx - (m + l\beta t)y + nz] \right\} \tag{14}$$

where k is an arbitrary constant. This gives, when $t = 0$,

$$\text{v} = v_0 = \frac{\mp k}{l^2 + m^2 + n^2} \sin my \begin{matrix}\sin \\ \cos\end{matrix} (lx + nz), \tag{15}$$

which fulfils (6) if $\sin mb = 0$, and allows us, by proper summation, for the different admissible values of m, and summation or integration with reference to l and n, with properly determined values of k, after the manner of Fourier, to give any arbitrarily assigned initial value to v for every value of x, y, z from $x = -\infty$ to $x = +\infty$, $y = 0$ to $y = b$, and $z = -\infty$ to $+\infty$. The same summation and integration applied to (13) gives v for all values of x, y, z, t.

It remains to find the value of $\mathfrak{v}$ which must satisfy (2), (7), (8). To do this Lord Kelvin first finds a real (simple harmonic) periodic solution of (2), fulfilling the conditions

$$\left.\begin{aligned} v &= C\cos\omega t + D\sin\omega t \\ \frac{dv}{dy} &= C'\cos\omega t + D'\sin\omega t \end{aligned}\right\} \text{ when } y = 0, \tag{16}$$

$$\left.\begin{aligned} v &= \mathfrak{C}\cos\omega t + \mathfrak{D}\sin\omega t \\ \frac{dv}{dy} &= \mathfrak{C}'\cos\omega t + \mathfrak{D}'\sin\omega t \end{aligned}\right\} \text{ when } y = b, \tag{17}$$

where $C, D, C', D', \mathfrak{C}, \mathfrak{D}, \mathfrak{C}', \mathfrak{D}'$ are eight assigned arbitrary functions of x, z. Then, by taking $\int_0^\infty d\omega f(\omega)$ of each of these after the manner of Fourier, one solves the problem of determining the motion produced throughout the fluid, by giving to every point of its plane boundaries an infinitesimal displacement, of which each of the three components is an arbitrary function of x, z, t.* Lastly, by taking these functions each $= 0$, from $t = -\infty$ to $t = 0$, and each equal to minus the value of v or $d\text{v}/dy$, as the case may be, for every point of each boundary, when $t > 0$, we find $\mathfrak{v}$ of equations (2), (3), (7), (8).

* As far as v is concerned we have only to deal with arbitrary boundary-values of v and of dv/dy, the latter being obtained from those of u, w by the equation of continuity.

The value of v satisfying (2), (3), (16), (17) is obtained by first finding an imaginary type solution.* Assume

$$v = e^{i(\omega t + lx + nz)}\, V \tag{18}$$

$$\sigma = e^{i(\omega t + lx + nz)} S. \tag{19}$$

Equation (2) then becomes

$$\frac{d^2 S}{dy^2} = \left(l^2 + n^2 + \frac{i(\omega + l\beta y)}{\nu}\right) S. \tag{20}$$

This may be solved by series proceeding in ascending powers of

$$l^2 + n^2 + i(\omega + l\beta y)/\nu$$

which are seen to be essentially convergent for all values. The form of S having thus been found, the solution of (2) can be expressed by using integral forms, and it involves four arbitrary constants; by the aid of these arbitrary constants, any prescribed values can be given to v and to dv/dy for $y = 0$ and $y = b$. Thus a real value of v satisfying (2), (3), (16), (17) may be obtained.

Now, the v solution, expressed by (13), comes essentially to nothing asymptotically as time advances. Hence, Lord Kelvin states, the $\mathfrak{v}$ of (2), (3), (7), (8), which rises gradually from zero at $t = 0$, comes asymptotically to zero again. He concludes that the steady motion is stable.

ART. 3. *His Solution of the Second Problem and its modification to suit the First Problem.*

In the second paper, which, as stated above, deals with the case in which the steady velocity is expressed by a quadratic function of y, Lord Kelvin writes as in (18), above,

$$v = e^{i(\omega t + lx + nz)}\, V,$$

and obtains the differential equation satisfied by V, which is of the fourth order. He shows how four independent solutions of it may be obtained in the form of series in ascending powers of y, convergent for all values of y, unless ν be zero. The rest of his discussion is by no means full; I trust I do not misinterpret it in the following statements. He appears to indicate that by means of the four arbitrary constants which occur in the value of V, any values desired can be assigned to V and to dV/dy for $y = 0$ and $y = b$, and that by integration or summation with respect to ω, l, n, one can thus obtain the motion produced in the fluid by giving the plane boundaries $y = 0$, $y = b$,

* At this stage of Lord Kelvin's work, in his equation (49), there occurs an error which is noted in an "erratum" prefixed to the bound volume of the Phil. Mag.

displacements which are arbitrary functions of x, z, t, indicating in a footnote that this same method may be used as affording a complete discussion of the former problem without any introduction of the v which satisfies (2), (3), (6). He states that the essential convergence of these series proves that the steady motion is stable, however small be ν, provided that it is not zero.

If ν be zero, the series become divergent in a certain region, thus giving rise to the "disturbing infinity" alluded to in Part I., Chap. I., p. 19.

ART. 4. *Lord Rayleigh's Criticism of the latter Solution.*

Commenting on these investigations, Lord Rayleigh writes*—"... I must confess that the argument does not appear to me demonstrative. No attempt is made to determine whether in free disturbances of the type e^{int} (in his notation $e^{i\omega t}$) the imaginary part of n is finite, and if so whether it is positive or negative. If I rightly understand it, the process consists in an investigation of forced vibrations of arbitrary (real) frequency, and the conclusion depends upon a tacit assumption that if these forced vibrations can be expressed in a periodic form, the steady motion from which they are deviations cannot be unstable. A very simple case suffices to prove that such a principle cannot be admitted. The equation to the motion of the bob of a pendulum situated near the highest point of its orbit is

$$d^2x/dt^2 - m^2x = X,$$

where X is an impressed force. If $X = \cos pt$, the corresponding part of x is

$$x = -\frac{\cos pt}{p^2 + m^2};$$

but this gives no indication of the inherent instability of the situation expressed by the free 'vibrations,'

$$x = Ae^{mt} + Be^{-mt}."$$

This criticism is evidently directed against the argument in the second of the two papers to which I have referred.

ART. 5. *Lord Rayleigh's Remarks on the Special Solution.*

In a later paper Lord Rayleigh, referring evidently to Lord Kelvin's first investigation, wrote†:—

"... In the particular case where the original vorticity is uniform, the problem of small disturbances has been solved by Lord Kelvin, who shows

* "On the question of the Stability of the Flow of Fluids," Phil. Mag., xxxiv., 1892, p. 67. Collected Papers, iii., p. 582.

† "On the Stability or Instability of certain Fluid Motions," Proc. Lond. Math. Soc. xxvii., 1895; Collected Papers, iv., p. 209.

that the motion is stable by the aid of a special solution not proportional to a simple exponential function of the time. If we retain the supposition of the present paper that the disturbance as a function of the time is proportional to e^{int}, we obtain an equation [(52) in Lord Kelvin's paper] which has been discussed by Stokes. From his results it appears that it is not possible to find a solution applicable to an unlimited fluid which shall be periodic with respect to x, and remain finite when $y = \pm \infty$, and this whether n be real or complex. The cause of the failure would appear to lie in the fact indicated by Lord Kelvin's solution, that the stability is ultimately of a higher order than can be expressed by any simple exponential function of the time."

ART. 6. *No Proof of Stability in either Solution. A tacit Assumption in the special one.*

Lord Rayleigh's objection to the argument in Lord Kelvin's latter paper appears unanswerable. The precise point of failure in the solution is that it does not in reality satisfy the most general conditions which may be assigned, just as, in the problem of the pendulum which Lord Rayleigh instances, the most general conditions cannot be satisfied without the introduction of the terms

$$Ae^{mt} + Be^{-mt}.$$

When the values of v and dv/dy are assigned at the bounding planes for all values of x, z, t, Lord Kelvin's solution is evidently an absolutely determinate one; but the initial state of things in the interior may be arbitrarily prescribed; and to allow this to be done there must evidently be added solutions which make v and dv/dy always zero at the bounding places: in other words, free disturbances.

Now, the special solution which Lord Rayleigh accepts in the second passage quoted (Art. 5), contains no reference to the free disturbances any more than does the solution which he rejects; and, on examination, it must, I think, be held that neither does it afford a proof of the stability of the motion. The value of $\mathfrak{v}$ in it, like that of v in the other, is completely determined by the boundary conditions (8) without any reference to the initial condition (7); and the statement in the penultimate sentence of Lord Kelvin's first investigation that $\mathfrak{v}$ rises gradually *from zero* at $t = 0$ thus involves an unjustified assumption that the solution which satisfies (2), (3), (8) will satisfy (7) also.

Art. 7. *The Assumption is valid, if Steady Motion exponentially Stable; not if exponentially Unstable.*

On consideration, it appears that this assumption may be shown to be correct, provided the free disturbances have stability of the ordinary exponential character; but that it would be incorrect if, for instance, any of them were exponentially unstable or neutral; this being so, the argument begs the question at issue. For, if a system in an exponentially stable state, whether of equilibrium or motion, be subjected to a simply harmonic disturbing force, (or motion affecting a definite coordinate), of any definite period, the solution in which the disturbance is simply harmonic and of the same period is known to become asymptotically correct as the time increases indefinitely, whatever may be the initial conditions (at least if the number of coordinates is finite). When the disturbing force is expressed as a Fourier integral, each element of which is simply periodic in time, and the elementary periodic disturbances which correspond to each in the fashion just described are combined by integration, it seems reasonable to infer that a similar statement would hold good for the resulting integral disturbance. When the range of time through which this resolution of the disturbing force is effected extends (say) from $-t_0$ to $+\infty$, then, at any instant, t, this force has been in operation for a time $t + t_0$, even though it may have been zero through a great portion of this interval, and accordingly the solution obtained in this manner is, if the state be exponentially stable, sufficiently accurate, provided $t + t_0$ is sufficiently great, whatever may have been the disturbance (supposed finite) at the time $-t_0$. But if the disturbing force is zero from $t = -t_0$ to $t = 0$, then if the state is exponentially stable, and t_0 is great enough, whatever finite disturbance may exist at the time $-t_0$, it must be sensibly reduced to zero at $t = 0$; so that in this mode of procedure we do, indeed, obtain the solution in which there is no disturbance at the time zero. We have only to suppose t_0 increased indefinitely to obtain the case with which we have here to deal; and hence it appears that the value of $\mathfrak{v}$ determined from (2), (3), (8) does indeed satisfy (7). But this argument fails, unless it is *known* that the state is exponentially stable.

Art. 8. *Mathematical Investigation of a simple example illustrating Validity of this Objection.*

A simple instance of many which could be cited in which the analysis is simple may serve to illustrate the argument, and especially to show that the result need not hold for an unstable state; the elaboration of a formal proof applicable to a case in which the number of independent coordinates is infinite would probably be a problem of considerable difficulty. Consider a system possessing only one coordinate, and governed by the equation

$$d^2x/dt^2 + (a+b)\,dx/dt + abx = X, \tag{21}$$

where, when t is negative, X is zero, and, when t is positive, $X = e^{-ct}$, c being positive, or having its real part positive. The solution in which at $t = 0$ x and dx/dt are zero, is known to be, for positive values of t,

$$(a-b)(b-c)(c-a)\,x = (b-a)\,e^{-ct} + (c-b)\,e^{-at} + (a-c)e^{-bt}. \tag{22}$$

By means of the equation

$$f(t) = \pi^{-1}\int_0^\infty d\omega \int_{-\infty}^\infty f(u)\cos\omega\,(u-t)\,du, \tag{23}$$

Fourier analysis of the disturbing force gives

$$e^{-ct} = \pi^{-1}\int_0^\infty \frac{c\cos\omega t + \omega\sin\omega t}{c^2+\omega^2}\,d\omega. \tag{24}$$

The solution of

$$d^2x/dt^2 + (a+b)\,dx/dt + abx = c\cos\omega t + \omega\sin\omega t, \tag{25}$$

which is of the same period as the disturbing force, being

$$(b-a)x = \frac{(ac-\omega^2)\cos\omega t + (a+c)\,\omega\sin\omega t}{a^2+\omega^2} - \frac{(bc-\omega^2)\cos\omega t + (b+c)\,\omega\sin\omega t}{b^2+\omega^2}, \tag{26}$$

the integral solution obtained in the way indicated is accordingly

$$(b-a)\,\pi x = \int_0^\infty \frac{(ac-\omega^2)\cos\omega t + (a+c)\,\omega\sin\omega t}{(a^2+\omega^2)(c^2+\omega^2)}\,d\omega - \int_0^\infty \frac{(bc-\omega^2)\cos\omega t + (b+c)\,\omega\sin\omega t}{(b^2+\omega^2)(c^2+\omega^2)}\,d\omega,$$

or

$$(a-b)(b-c)(c-a)\pi x = (b-a)\int_0^\infty \frac{c\cos\omega t + \omega\sin\omega t}{c^2+\omega^2}\,d\omega + (c-b)\int_0^\infty \frac{a\cos\omega t + \omega\sin\omega t}{a^2+\omega^2}\,d\omega + (a-c)\int_0^\infty \frac{b\cos\omega t + \omega\sin\omega t}{b^2+\omega^2}\,d\omega. \tag{27}$$

The first integral on the right is zero when t is negative, and πe^{-ct} when t is positive; if the real part of a is positive, the second integral is zero when t is negative, and πe^{-at} when t is positive; but, on the other hand, if the real part of a is negative, it is zero when t is positive, and πe^{-at} when t is negative;* while it is infinite if the real part of a is zero; and similar statements hold for the third term. Thus the value of x as given by (27) agrees with the correct

* These statements are equivalent to equation (24), a and c being interchanged where necessary.

value given in (22) if the real parts of a, b are both positive, but not if either or both are negative or zero.

A system subject to an equation of the type

$$dx/dt + ax = X$$

affords a still simpler illustration, and might be held to be more appropriate to the problem in view.

ART. 9. *Other Objections to the special Solution as a Proof of Stability.*

The same penultimate sentence of Lord Kelvin's investigation also contains another unproven assumption, viz.: that $\mathfrak{v}$ comes asymptotically to zero as t increases to ∞. This statement, like the preceding one (*i.e.*, that it rises gradually from zero at $t = 0$), is only known to be true for the boundary values of $\mathfrak{v}$. This objection to the second statement may be expressed as follows:—In the first place, the fact that the value of v, simply-periodic in time which satisfies (2), (3), (16), (17), can be expanded in a convergent series of powers of y, does not preclude the impossibility of so choosing ω, l, m, n, that v could, through some portion of the interior, be made very great, or even as great as we please, compared with its values at the boundaries;* and in the second, the mere fact that the resultant value of $\mathfrak{v}$ is obtained as the integral effect of such solutions corresponding to different values of ω, when viewed in the light of the known possibilities of Fourier analysis, so far from showing that it eventually diminishes indefinitely, is seen to impose no limit whatever on its value.

Again, the tacit assumption that, if the steady motion is stable for disturbances in which v varies as $\sin my$, it is also stable for those of a more general type, appears to require justification.

ART. 10. *The Special Solution contains a Proof that the Motion, if rapid enough, will be practically Unstable. Two Modifications of the Solution partially satisfying the Boundary-Conditions.*

Thus, Lord Kelvin's special solution, equally with that included in his discussion of the more difficult problem, appears unacceptable as a proof of the stability of the steady motion. We have seen, however, that if it be admitted, as will be proved in Chap. II. below, that the infinitesimal principal disturbances have stability of the ordinary simple exponential type, it does provide an investigation of the propagation of an arbitrary initial

* It may be held that this remark, if it stood alone, would not affect Lord Kelvin's inference that the steady motion is stable if the initial disturbance be of the type he chooses and *sufficiently small.*

disturbance. And although the function $\mathfrak{v}$ of equations (2), (7), (8) is not easily obtainable in a form which enables us to calculate numerical values, important conclusions may be drawn from the form which this solution gives for v without any regard whatever to $\mathfrak{v}$. Whether the infinitesimal disturbances are stable or not, it furnishes, in fact, a proof that the motion may be practically unstable, and shows qualitatively, and to some extent quantitatively, the circumstances in which instability may be expected. (In short, I cannot make any substantial advance in the matter of showing that there *will* be instability beyond pointing out what may be inferred from this solution.) There is good reason for supposing that, if lb, mb, nb are large, the precise conditions which prevail at the boundaries cannot modify the disturbance appreciably at any sensible distance, and thus cannot much affect the question of stability for disturbances of small wave-lengths in the x and z directions. It is seen that, if the viscosity is sufficiently small, just as when it is altogether neglected,* the initial disturbance may, owing to the expression $l^2 + (m - l\beta t)^2 + n^2$ in the denominator of v, as given by (13), (14), increase very much in spite of the exponential multiplier. We may, moreover, easily amend the expression for v, by adding to it the proper solution of the equation $\nabla^2 v = 0$, so as to obtain a solution which shall satisfy either of the boundary-conditions $v = 0$, $dv/dy = 0$, but not both.† If we select the former alternative, such a solution corresponding to an initial disturbance in which

$$v = v_0 = B \cos lx \sin my \cos nz \tag{28}$$

is

$$\frac{2v \sinh \lambda b}{(\lambda^2 + m^2) B \cos nz} = \frac{Exp\,\{-\nu t\,(\lambda^2 + m^2 - lm\beta t + l^2\beta^2 t^2/3)\}}{\lambda^2 + (m - l\beta t)^2}$$

$$\times \{\sinh \lambda b \sin[lx + (m - l\beta t)\,y] - \sinh\lambda\,(b - y) \sin lx - \sinh \lambda y \sin[lx + (m - l\beta t)\,b]$$

$$- \frac{Exp\,[-\nu t(\lambda^2 + m^2 + lm\beta t + l^2\beta^2 t^2/3]}{l^2 + (m + l\beta t)^2}$$

$$\times \{\sinh \lambda b \sin[lx - (m + l\beta t)y] - \sinh\lambda\,(b - y) \sin lx - \sinh \lambda y \sin[lx - (m + l\beta t)\,b]\}, \tag{29}$$

in which $\lambda^2 = l^2 + n^2$. The solution in the case of a two-dimensioned disturbance, in which $n = 0$, $\lambda = l$, can be completed by writing down the

* Compare Part I., Arts. 4–8, with Arts. 10, 11 here.

† Of course, Lord Kelvin's typical initial disturbance of (15) violates the boundary condition $dv/dy = 0$; the conditions $v = 0$, $d^2v/dy^2 = 0$ are somewhat simpler; but even in that case I cannot complete the solution in a form which gives results suitable for quantitative comparisons.

corresponding value of u. It is

$$\frac{2lu \sinh lb}{(l^2+m^2)B} = \frac{Exp\,\{-\nu t(l^2+m^2-lm\beta t+l^2\beta^2t^2/3)\}}{l^2+(m-l\beta t)^2}$$

$$\times\{-(m-l\beta t)\sinh lb \sin[lx+(m-l\beta t)y]+l\cosh l(b-y)\cos lx - l\cosh ly\cos[lx+(m-l\beta t)b]\}$$

$$-\frac{Exp\,[-\nu t(l^2+m^2+lm\beta t+l^2\beta^2t^2/3)]}{l^2+(m+l\beta t)^2}$$

$$\times\{(m+l\beta t)\sinh lb\sin[lx-(m+l\beta t)y]+l\cosh l(b-y)\cos lx - l\cosh ly \cos[lx-(m+l\beta t)b]\}. \quad (30)$$

It is seen that these expressions differ from those obtained when viscosity is ignored (Part I., equations (28), p. 26; (38), p. 28;) only by the presence of the exponential multipliers, and become identical with them if ν is equated to zero. There thus appears to be no necessity for the suggestion thrown out by Lord Rayleigh that, in these questions of stability, investigations in which viscosity is altogether ignored may possibly be inapplicable to the limiting case of a viscous fluid when the viscosity is supposed infinitely small.*

ART. 11. *For suitable Values of Constants in First Modification the Disturbance will Increase greatly. Substitution of a numerical Value suggested by Experiment.*

Taking then the values of u, v given by (29), (30), they are derivable from a stream function, ψ, given by

$$\frac{2l\psi \sinh lb}{(l^2+m^2)B} = Exp\,[-\nu t(l^2+m^2-lm\beta t+l^2\beta^2t^2/3]$$

$$\frac{\sinh lb\cos[lx+(m-l\beta t)y]-\sinh l(b-y)\cos lx-\sinh ly\cos[lx+(m-l\beta t)b]}{l^2+(m-l\beta t)^2}$$

– another term derivable by changing the sign of m. (31)

Here

$$\frac{-2l\nabla^2\psi}{(l^2+m^2)B} = Exp\,[-\nu t(l^2+m^2-lm\beta t+l^2\beta^2t^2/3].\cos[lx+(m-l\beta t)y]$$

– another term derivable by changing the sign of m. (32)

If T be the average energy of the relative motion per unit length of stream,

$$4T\pi/l = -\int_0^{2\pi/l}\int_0^b \psi\nabla^2\psi dx dy. \quad (33)$$

Making use of this, on performing the integrations, and comparing the value

* "On the Question of the Stability of the Flow of Fluids," Phil. Mag., xxxiv., p. 61, p. 67, 1892; Scientific Papers, iii., p. 577, p. 582.

of T thus found with its initial value, i.e., $T_0 = B^2 b\,(l^2 + m^2)/8l^2$, there results

$$\frac{T}{T_0} = \frac{(l^2+m^2)}{2}\left[\frac{Exp[-2\nu t(l^2+m^2-lm\beta t+l^2\beta^2t^2/3)]}{l^2+(m-l\beta t)^2} + \frac{Exp[-2\nu t(l^2+m^2+lm\beta t+l^2\beta^2t^2/3)]}{l^2+(m+l\beta t)^2}\right.$$

$$-\left\{\frac{Exp\,[-\nu t(l^2+m^2-lm\beta t+l^2\beta^2t^2/3)]}{l^2+(m-l\beta t)^2} - \frac{Exp\,[-\nu t(l^2+m^2+lm\beta t+l^2\beta^2t^2/3)]}{l^2+(m+l\beta t)^2}\right\}^2$$

$$\left.\times\, l^2\,\frac{\cosh lb - \cos(m - l\beta t)\,b}{\tfrac{1}{2}lb\sinh lb}\right]. \tag{34}$$

As the terms to be subtracted from the two first are essentially positive, there is no possibility of any great increase, unless the first two are large; and even in the absence of the exponential factors, this can occur only if m/l is large, and then solely during the time in which $m - l\beta t$ is of order not larger than l. At such times, the terms which have $l^2 + (m + l\beta t)^2$ in the denominator may be neglected in comparison with the others. During such a time, if m/l is large, we may approximately replace* the exponential factor of the two remaining terms by $Exp\,\{-2\nu m^3/(3l\beta)\}$, and thus obtain

$$\frac{T}{T_0} \doteqdot e^{\frac{-2\nu m^3}{3l\beta}}\cdot\frac{m^2}{2l^2}\left\{1 - \frac{l^2}{l^2 + (m - l\beta t)^2}\;\frac{\cosh lb - \cos(m - l\beta t)\,b}{\tfrac{1}{2}lb\sinh lb}\right\}. \tag{35}$$

As the last factor is less than unity, a large value for T/T_0 requires that $\nu m^3/(l\beta)$ should not be large, i.e. that $m^2b^2 . m/l$ should not be large compared with $\beta b^2/\nu$; now the smallest possible value of mb is π, and m/l is large, so that instability requires $\beta b^2/\nu$ to be large. Conversely, if m/l, mb are large, and $\beta b^2/\nu$ large enough to be of the same order as $m^2b^2 . m/l$, an initial disturbance of the type given by (28), and subject to the boundary-conditions required by (29), (30), one of which is $v = 0$, will increase very much before dying out. At the time when $m - l\beta t = 0$, we have in fact

$$\frac{T}{T_0} \doteqdot e^{\frac{-2\nu m^3}{3l\beta}}\cdot\frac{m^2}{2l^2}\left\{1 - \frac{\tanh\tfrac{1}{2}lb}{\tfrac{1}{2}lb}\right\}. \tag{36}$$

With the relative magnitudes chosen for the constants, the exponential factor is not small, and the product of the other factors is large, its approximate values in the extreme cases of lb large and of lb small being respectively $m^2/2l^2$ and $m^2b^2/24$.

It may be of interest to take values of the constants for which a somewhat similar motion has been found experimentally to be unstable, and ascertain to some extent how much they would allow a disturbance of the type (29), (30) to increase. Couette found† that, when a cylinder of radius 14·6395 cm.

* I.e. in the sense that this gives the *index* of the exponential factor with sensible accuracy.

† "Etudes sur le frottement des liquides," Annales de Chimie et de Physique, (6) xxi., p. 433.

was rotated in water at 16·7° C. outside a concentric one of radius 14·3930 cm., the motion ceased to be thoroughly stable when the speed exceeded about 56 revolutions per minute; taking ν to be ·011, this corresponds to a value of $\beta b^2/\nu$ which is about 1940. Writing $\beta b^2/\nu = 1900$, it is seen that the disturbance could not increase greatly. Going back to (34), but writing $m - l\beta t = 0$, and retaining only the terms which are more important, we have

$$\frac{T}{T_0} \doteqdot Exp\left\{\frac{-2b^2m(3l^2+m^2)}{5700l}\right\}.\frac{l^2+m^2}{2l^2}\left\{1-\frac{\tanh\frac{1}{2}lb}{\frac{1}{2}lb}\right\}. \tag{37}$$

The final factor is less than unity, and also less than $l^2b^2/12$; thus its value is less than

$$Exp\left\{\frac{-2mb}{5700lb}\left(3b^2l^2+m^2b^2\right)\right\}.\frac{l^2b^2+m^2b^2}{2l^2b^2}, \tag{38}$$

and also less than

$$Exp\left\{\frac{-2mb}{5700lb}\left(3l^2b^2+m^2b^2\right)\right\}.\frac{l^2b^2+m^2b^2}{24}. \tag{39}$$

For either of these expressions to be a maximum, there is required

$$(m^2b^2+l^2b^2)^2 = 1900lb.mb, \tag{40}$$

or, if m/l is supposed large

$$m^3b^3 \doteqdot 1900lb; \tag{41}$$

then the former becomes approximately

$$e^{-\frac{2}{3}}\left(\frac{1}{2}+\frac{(1900)^2}{2m^4b^4}\right), \tag{42}$$

and the latter

$$e^{-\frac{2}{3}}\left(\frac{m^2b^2}{24}+\frac{m^6b^6}{24(1900)^2}\right) \tag{43}$$

A superior limit to (37) is thus the smaller of (42), (43), and thus their common value, when they are equal, i.e., about 15. The maximum value of (37) appears in fact to be about 4; and it approaches this value when $lb = 2$, $mb = 5\pi$.

It may be seen that, for this value of $\beta b^2/\nu$, the terms omitted from (34) are unimportant, and that the approximations used give nearly its maximum value and the time at which that occurs.

If the disturbance were taken alone which involves the first exponential factor in (31), (32), somewhat similar results would be obtained as to the possibilities of its increase.

ART. 12. *A similar Investigation for the Second Modification.*

If we take the solution which would make dv/dy, and therefore u, zero, instead of v, at the bounding planes, it is seen that the two-dimensioned form corresponding to an initial disturbance in which

$$v = v_0 = B \sin lx \cos my \tag{44}$$

has a stream function given by

$$\frac{2l\psi \sinh lb}{(l^2 + m^2) B} = \frac{Exp\left[-\nu t\left(l^2 + m^2 - lm\beta t + l^2\beta^2 t^2/3\right)\right]}{l^2 + (m - l\beta t)^2}$$
$$\times \{\sinh lb \cos\left[lx + (m - l\beta t) y\right] - l^{-1}(m - l\beta t) \cosh l(b - y) \sin lx$$
$$+ l^{-1}(m - l\beta t) \cosh ly \sin\left[lx + (m - l\beta t) b\right]\}$$
$$+ \text{another term derivable by changing the sign of } m. \tag{45}$$

In this case, the ratio of increase at time t is

$$\frac{T}{T_0} = \frac{l^2 + m^2}{2}\left[\frac{Exp\left[-2\nu t(l^2+m^2-lm\beta t+l^2\beta^2t^2/3)\right]}{l^2+(m-l\beta t)^2} + \frac{Exp\left[-2\nu t(l^2+m^2+lm\beta t+l^2\beta^2t^2/3)\right]}{l^2+(m+l\beta t)^2}\right.$$
$$+\left\{\frac{(m-l\beta t)Exp\left[-\nu t(l^2+m^2-lm\beta t+l^2\beta^2t^2/3)\right]}{l^2+(m-l\beta t)^2} - \frac{(m+l\beta t)Exp\left[-\nu t(l^2+m^2+lm\beta t+l^2\beta^2t^2/3)\right]}{l^2+(m+l\beta t)^2}\right\}^2$$
$$\left.\times \frac{\cosh lb - \cos(m - l\beta t) b}{\frac{1}{2} lb \sinh lb}\right]. \tag{46}$$

Here, again, there is a possibility of a large increase if m/l is large.* At the instant when $m - l\beta t$ is zero, the only term in this which is not negligible assumes the form

$$Exp\left\{\frac{-2\nu m}{3l\beta}(3l^2 + m^2)\right\} \cdot \frac{l^2 + m^2}{2l^2} \tag{47}$$

simpler than (37), and capable of assuming a much greater value. A condition that (47) should be a maximum is

$$\nu (l^2 + m^2)^2 = lm\beta,$$
$$\text{or} \quad \nu m^3 \doteqdot l\beta; \tag{48}$$

and then it is approximately

$$e^{-\frac{2}{3}}\beta^2/2m^4\nu^2, \quad \text{or} \quad e^{-\frac{2}{3}}b^4\beta^2\nu^{-2}/2m^4b^4. \tag{49}$$

If $\beta b^2/\nu = 1900$ and mb has its lowest value, π, this is nearly 9500. Taking $\beta b^2/\nu = 1940$, we have in round numbers 10000.

* It is not evident that, as in the case of the first modification, there is no possibility of a great increase under any other circumstances.

If we took the disturbance indicated by the first term alone of ψ in (45), almost the same result would be obtained.

The difference between these two solutions, and between their results as to stability, strengthens the view that boundary-conditions are unimportant if, and only if, lb is large. It is not suggested that when instability actually occurs, the increase in a disturbance is as small as that obtained in the former solution, or as great as that in the latter. The boundary-conditions to which they refer are not those which occur in the experiment; lb is not large (in the latter solution, very small), so that the violation of boundary-conditions is important; and even the initial disturbance does not satisfy the realizable boundary-conditions.

CHAPTER II.

THE FUNDAMENTAL FREE DISTURBANCES OF A STREAM WHICH IS SHEARING UNIFORMLY.

ART. 13. *The Period-Equation for the Boundary-Conditions* $\nabla^2 v = 0$.

In a passage quoted above,* Lord Rayleigh appears to suggest that possibly in the case of a stream of uniform vorticity there may not be free disturbances which involve the time in the usual exponential or trigonometrical form, i.e. varying as e^{pt}, where p is a real or complex constant. I proceed to consider this question. Referring to Lord Kelvin's analysis given in Chapter I., if in equation (20) of that Chapter, we write $\omega i = p$, it assumes the form

$$d^2S/dy^2 = \{l^2 + n^2 + (p + il\beta y)/\nu\}S, \tag{1}$$

where

$$S = (d^2/dy^2 - l^2 - n^2)V. \tag{2}$$

The solutions of (1) are given by Lord Kelvin in the form of infinite series; and the equation had previously been discussed by Stokes† and others. The solution in fact is, if we replace $l^2 + n^2$ by λ^2,

$$S = (\nu\lambda^2 + p + l\beta yi)^{\frac{1}{2}}$$
$$\times \left\{AI_{\frac{1}{3}}\left[\frac{2}{3}\left\{\frac{l\beta}{\nu}\left(-yi - \frac{\nu\lambda^2 + p}{l\beta}\right)^3\right\}^{\frac{1}{2}}\right] + BI_{-\frac{1}{3}}\left[\frac{2}{3}\left\{\frac{l\beta}{\nu}\left(-yi - \frac{\nu\lambda^2 + p}{l\beta}\right)^3\right\}^{\frac{1}{2}}\right]\right\}. \tag{3}$$

where I_n is the function connected with the Bessel function J_n by the relation

$$I_n(\theta) = i^{-n}J_n(i\theta) = \frac{\theta^n}{2^n\Pi(n)}\left\{1 + \frac{\theta^2}{2.(2n+2)} + \frac{\theta^4}{2.4.(2n+2)(2n+4)} + \ldots\right\}. \tag{4}$$

We may also write (3) in the form

$$S = A'\psi\left\{\left(\frac{l\beta}{\nu}\right)^{\frac{1}{3}}\left(-yi - \frac{\nu\lambda^2 + p}{l\beta}\right)\right\} + B'\phi\left\{\left(\frac{l\beta}{\nu}\right)^{\frac{1}{3}}\left(-yi - \frac{\nu\lambda^2 + p}{l\beta}\right)\right\}, \tag{5}$$

where

$$\psi(Y) = Y + \frac{Y^4}{3.4} + \frac{Y^7}{3.4.6.7} + \ldots, \tag{6}$$

$$\phi(Y) = 1 + \frac{Y^3}{2.3} + \frac{Y^6}{2.3.5.6} + \ldots. \tag{7}$$

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* See Art. 5, p. 85.

† It was in connexion with this equation that Stokes published his investigation of the asymptotic expansion of Bessel's functions; "On the Numerical Calculation of Definite Integrals and Infinite Series," Trans. Camb. Phil. Soc., ix., Part i., 1850; Math. and Phys. Paper ii, p. 329.

The solution of (2), as an equation determining V, is easily expressible by means of integrals, and is so expressed by Lord Kelvin. He does not, however, make any reference to the problem of determining p so as to satisfy assigned boundary-conditions.

The most natural boundary-conditions to take would, of course, be that at each of the bounding-planes u, v, w should vanish; conditions which, as far as v is concerned, are equivalent to the vanishing of V and dV/dy. The analysis would obviously be much simplified, however, if two of the four conditions which V can satisfy should be the vanishing of S at each of the planes; and it will be chiefly this case that I shall consider. It is readily seen that we should have this case if the boundary-conditions were that v should vanish, and that the tangential forces on the bounding planes should be the same in the disturbed as in the steady motion.

Denoting the bounding-planes by $y = \pm a$, instead of $y = 0, b$, as in Part I., Chap. I., the equation determining the value of p evidently takes the form

$$I_{\frac{1}{3}}\left[\frac{2}{3}\left\{\frac{l\beta}{\nu}\left(-\frac{\nu\lambda^2+p}{l\beta}+ai\right)^3\right\}^{\frac{1}{2}}\right] I_{-\frac{1}{3}}\left[\frac{2}{3}\left\{\frac{l\beta}{\nu}\left(-\frac{\nu\lambda^2+p}{l\beta}-ai\right)^3\right\}^{\frac{1}{2}}\right]$$
$$-I_{-\frac{1}{3}}\left[\frac{2}{3}\left\{\frac{l\beta}{\nu}\left(-\frac{\nu\lambda^2+p}{l\beta}+ai\right)^3\right\}^{\frac{1}{2}}\right] I_{\frac{1}{3}}\left[\frac{2}{3}\left\{\frac{l\beta}{\nu}\left(-\frac{\nu\lambda^2+p}{l\beta}-ai\right)^3\right\}^{\frac{1}{2}}\right] = 0. \quad (8)$$

As the form of this is unaltered by changing the sign of i, complex roots occur in pairs in the usual fashion.

ART. 14. *This Period Equation has an Infinite Number of Roots.*

In view of the suggestion of Lord Rayleigh,* referred to above, it seems desirable to prove, in the first place, that this equation in p has an infinite number of roots; it has, in fact, an infinite number whose real parts are negative. This may be shown by the aid of the approximate expressions for the I functions for large values of the parameter. If we suppose that $(\nu\lambda^2+p)/l\beta$ has its real part negative, large compared with its imaginary part, and large compared with a, we may take the argument of

$$\left(-\frac{\nu\lambda^2+p}{l\beta}+ai\right)^{\frac{3}{2}}$$

to be a small positive angle, and that of

$$\left(-\frac{\nu\lambda^2+p}{l\beta}-ai\right)^{\frac{3}{2}}$$

* See Art. 5, p. 85.

to be a small negative angle. Now, if the argument of x lies between the limits $\pm\pi$, we have the equation*

$$I_{-k}(x) - I_k(x)$$
$$= (2/\pi x)^{\frac{1}{2}} \sin k\pi\, e^{-x}\left\{1 - \frac{(1-2k)(1+2k)}{8x} + \frac{(1-2k)(3-2k)(1+2k)(3+2k)}{8\,.\,16\,.\,x^2} + \ldots\right\} \tag{9}$$

in the sense that, provided $s - k + \frac{1}{2}$ is positive, the error in terminating the series on the right after the s^{th} term has a modulus less than that of the next term if the argument of x lies between the limits $\pm\pi/2$, and less than a certain multiple† of it if the argument of x lies between $\pi/2$ and π, or between $-\pi/2$ and $-\pi$. And, by writing in this equation $x = ye^{-\pi i}$, and dividing across by $\sin k\pi$, we obtain the equation

$$I_{-k}(y) + I_k(y) - i\cot ky\,\{I_{-k}(y) - I_k(y)\}$$
$$= (2/\pi y)^{\frac{1}{2}} e^{y}\left\{1 + \frac{(1-2k)(1+2k)}{8y} + \frac{(1-2k)(3-2k)(1+2k)(3+2k)}{8\,.\,16\,.\,y^2} + \ldots\right\}, \tag{10}$$

which holds in a sense obvious from the preceding sentence, provided the argument of y lies between 0 and 2π. While, by writing in (9), $x = ye^{+\pi i}$, there results

$$I_{-k}(y) + I_k(y) + i\cot ky\,\{I_{-k}(y) - I_k(y)\}$$
$$= (2/\pi y)^{\frac{1}{2}} e^{y}\left\{1 + \frac{(1-2k)(1+2k)}{8y} + \frac{(1-2k)(3-2k)(1+2k)(3+2k) + \ldots}{8\,.\,16\,.\,y^2}\right\}, \tag{11}$$

provided the argument of y lies between 0 and -2π.

Thus, if y is large, and its argument lies between $\pm\pi/2$, it follows from (9) that the term involving $I_{-k}(y) - I_k(y)$, which occurs in the left-hand members of (10), (11), may be neglected, so that within these limits for large values of y we have the approximate equations

$$I_{-k}(y) - I_k(y) \doteqdot (2/\pi y)^{\frac{1}{2}} \sin k\pi\, e^{-y}, \tag{12}$$

$$I_{-k}(y) + I_k(y) \doteqdot (2/\pi y)^{\frac{1}{2}} e^{y}. \tag{13}$$

Accordingly, if $AI_k(x) + B_{-k}(x)$ is to vanish for two large values of x, whose arguments lie between $\pm\pi/2$, the values must differ approximately by a multiple

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* "On the Product $J_m(x)\,J_n(x)$," Proc. Camb. Phil. Soc., x., Part III., equations (14), &c.; "On Divergent Hypergeometric Series," Trans. Camb. Phil. Soc., xvii., Part III., Art. 3, especially foot-notes, pp. 179–180; and Art. 11. In the foot-note on p. 179, for "$\pi \pm \gamma$" read "$\pm(\pi - \gamma)$". Some errata in Art. 11 are corrected in Vol. xix., Part I., p. 155.

† The multiplier depends on the argument of x, but not on the modulus.

of πi; and thus, if we make the further supposition, that the quantities

$$\left\{\frac{l\beta}{\nu}\left(-\frac{\nu\lambda^2+p}{l\beta}\pm ai\right)^3\right\}^{\frac{1}{2}}$$

are sufficiently large, equation (8), which expresses that S as given by (3) should vanish for two different values of the parameter, takes the approximate form

$$\frac{2}{3}\left\{\frac{l\beta}{\nu}\left(-\frac{\nu\lambda^2+p}{l\beta}+ai\right)^3\right\}^{\frac{1}{2}}-\frac{2}{3}\left\{\frac{l\beta}{\nu}\left(-\frac{\nu\lambda^2+p}{l\beta}-ai\right)^3\right\}^{\frac{1}{2}}\doteqdot r\pi i,$$

or

$$\frac{2}{3}\left\{\frac{l\beta}{\nu}\left(\frac{\nu\lambda^2+p}{l\beta}-ai\right)^3\right\}^{\frac{1}{2}}-\frac{2}{3}\left\{\frac{l\beta}{\nu}\left(\frac{\nu\lambda^2+p}{l\beta}+ai\right)^3\right\}^{\frac{1}{2}}\doteqdot r\pi, \qquad (14)$$

where r is any integer, positive, or negative.

If r is sufficiently large, whatever be the values of l, λ, this equation in p has one root such that the real part of $\nu\lambda^2+p$, and *a fortiori* the real part of p, is negative. (When the equation is rationalized, care must be taken to distinguish between it and the equation which would be obtained by connecting the two terms on the left-hand side by a plus instead of by a minus sign.) In fact, as we have already supposed that a is small compared with $(\nu\lambda^2+p)/l\beta$, the equation may be replaced by

$$2\left(\frac{l\beta}{\nu}\cdot\frac{\nu\lambda^2+p}{l\beta}\right)^{\frac{1}{2}}ai\doteqdot -r\pi,$$

giving

$$p\doteqdot -\nu(\lambda^2+r^2\pi^2/4a^2), \qquad (15)$$

a value which is wholly real and negative. The suppositions made in arriving at this approximate value of p, viz.: that $(\nu\lambda^2+p)/l\beta$ has its real part negative, large compared with its imaginary part, and large compared with a, and that

$$\frac{l\beta}{\nu}\left(\frac{\nu\lambda^2+p}{l\beta}\pm ai\right)^3$$

are sufficiently large, are accordingly justified, provided r is sufficiently large. And as r may be any integer if large enough, it thus appears that the approximate form of the period-equation has an infinity of roots.

Moreover, from the value found for p, it appears that by taking r large enough, the accurate form (8) of the period-equation may be represented as closely as we please by the approximate form (14), so that the actual period-equation must have an infinity of roots.

ART. 15. *Each Fundamental Disturbance satisfying the Boundary-Conditions* $\nabla^2 v = 0$ *is exponentially stable.*

It may next be shown that *all* the values of p which satisfy the period-equation (8) have a real negative part. This follows easily by a method which has been used by Lord Rayleigh in the discussion of similar questions when viscosity is ignored. The period-equation has been obtained by making the function S, which is a solution of equation (1), vanish for the two values $y = \pm a$. In equation (1), then, write $S = P + iQ$, $p = \theta + i\phi$, where P, Q, θ, ϕ are all real; separating the real and imaginary parts we have

$$\nu d^2P/dy^2 = (\nu\lambda^2 + \theta)P - (\phi + l\beta y)Q, \tag{16}$$

$$\nu d^2Q/dy^2 = (\nu\lambda^2 + \theta)Q + (\phi + l\beta y)P. \tag{17}$$

Multiplying the former by P, the latter by Q, and adding, we obtain

$$\nu(Pd^2P/dy^2 + Qd^2Q/dy^2) = (\nu\lambda^2 + \theta)(P^2 + Q^2). \tag{18}$$

Integrating with respect to y from $y = -a$ to $y = +a$, since S, and therefore both P and Q, vanish at the limits, we obtain

$$-\nu\int_{-a}^{+a}\{(dP/dy)^2 + (dQ/dy)^2\}dy = \int_{-a}^{+a}(\nu\lambda^2 + \theta)(P^2 + Q^2)dy. \tag{19}$$

The right-hand member must therefore be negative, so that not only must p have a negative real part, but that real part must be numerically greater than $\nu\lambda^2$.

If we multiply (17) by P, (16) by Q, and subtract, we obtain

$$\nu(Pd^2Q/dy^2 - Qd^2P/dy^2) = (\phi + l\beta y)(P^2 + Q^2). \tag{20}$$

Integrating with respect to y from $y = -a$ to $y = +a$, since P and Q both vanish at the limits, we obtain

$$0 = \int_{-a}^{+a}(\phi + l\beta y)(P^2 + Q^2)dy, \tag{21}$$

so that $\phi + l\beta y$ must change sign as y passes through some value between $-a$ and $+a$. Accordingly the value of ϕ must lie between the limits $\pm l\beta a$.

If the boundary-conditions assigned were that dS/dy should vanish at the bounding-planes, it may be readily seen that all the conclusions drawn above as to the existence of, and the nature of, the roots of the period-equation still hold.

While, if the boundary-conditions were that S should vanish at one bounding-plane, and dS/dy at the other, it may be seen that the period-equation has an infinity of roots, and that all the values of p have negative real parts numerically greater than $\nu\lambda^2$; the conclusion that the imaginary part of p lies between the limits $\pm l\beta ai$ would not, however, hold. And in the right-hand member of (14), $r\pi$ would be replaced by $(2r+1)\pi/2$, as we should now require, approximately, $Ae^x + Be^{-x}$ to vanish for one value of the parameter, and $Ae^x - Be^{-x}$ for another, so that the two values of the parameter would differ approximately by $(2r+1)\pi i/2$.

It thus appears that the fundamental modes of free disturbance possess stability of the ordinary simple exponential character, when the boundary-conditions include the vanishing of $\nabla^2 v$.

Art. 16. *For all values of l, n, there are an infinite number of Aperiodic Disturbances.*

Considering real values of p for which $\nu\lambda^2 + p$ is negative, if we take that value of

$$\left\{-\frac{\nu\lambda^2 + p}{l\beta} + yi\right\}^{\frac{3}{2}},$$

whose argument is zero when y is zero, then when y is a, its argument must lie between the limits 0 and $3\pi/4$;* and when y is $-a$, its argument must lie between 0 and $-3\pi/4$. Now, from (9), (10), there is one linear function of $I_{-k}(x)$ and $I_k(x)$, viz., a multiple of $I_{-k}(x) - I_k(x)$, which, for large values of x whose argument lies between $-3\pi/2$ and $+3\pi/2$, is approximately $x^{-\frac{1}{2}}e^{-x}$; and there is another function, viz., a multiple of $I_{-k}(x) + I_k(x)$, which, when the argument lies between 0 and π, assumes the approximate form

$$x^{-\frac{1}{2}}(e^x + i\cos k\pi \,.\, e^{-x}),$$

but which, when the argument lies between 0 and $-\pi$, is approximately

$$x^{-\frac{1}{2}}(e^x - i\cos k\pi \,.\, e^{-x}).$$

If, then, we write

$$\frac{2}{3}\left\{\frac{l\beta}{\nu}\left(-\frac{\nu\lambda^2+p}{l\beta} + ai\right)^3\right\}^{\frac{1}{2}} = u_1, \quad \frac{2}{3}\left\{\frac{l\beta}{\nu}\left(-\frac{\nu\lambda^2+p}{l\beta} - ai\right)^3\right\}^{\frac{1}{2}} = u_2, \qquad (22)$$

the period equation is, approximately,

$$\frac{e^{u_1} + i/2\,.\,e^{-u_1}}{e^{-u_1}} \doteq \frac{e^{u_2} - i/2\,.\,e^{-u_2}}{e^{-u_2}};$$

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* This is true for complex values also, since, as proved in Art. 15, the imaginary part of $\nu\lambda^2 + p$ lies between the limits $\pm\, l\beta ai$.

or, as it may be written,

$$- i\left(e^{u_1 - u_2} - e^{u_2 - u_1}\right) + e^{-u_1 - u_2} \doteqdot 0. \tag{23}$$

Substituting

$$u_1 = P + iQ, \quad u_2 = P - iQ, \tag{24}$$

this becomes

$$2 \sin 2Q + e^{-2P} \doteqdot 0. \tag{25}$$

Moreover, the form of (8) shows that when P is real, the accurate value of the left-hand member of (23) is a real quantity; and (10), (11) show that the errors in the expressions e^{u_1}, e^{-u_1} have moduli less than those of $Au_1^{-1}e^{u_1}$, $Bu_1^{-1}e^{-u_1}$, respectively; and those in e^{u_2}, e^{-u_2} have moduli less than those of $Au_2^{-1}e^{u_2}$, $Bu_2^{-1}e^{-u_2}$, respectively, where A, B are certain numbers. Thus the error in the left-hand member of (25) is less than

$$2\{(1 + AU^{-1})(1 + BU^{-1}) - 1\} + e^{-2P}\{(1 + BU^{-1})^2 - 1\}, \tag{26}$$

where U denotes the modulus of u_1 or u_2. And if

$$\frac{l\beta}{\nu}\left(-\frac{\nu\lambda^2 + p}{l\beta}\right)^3$$

is large enough, P, U can be made as great as ever we please. From this it is evident that, if

$$\frac{l\beta}{\nu}\left(-\frac{\nu\lambda^2 + p}{l\beta}\right)^3$$

is sufficiently great, on substituting a real value of p in the accurate expression for the left-hand member of (25), there is obtained a real magnitude which differs from $2 \sin 2Q$ by as little as ever we please. Consequently, for all values of l, λ, there are an infinite number of *real* negative values of p, given as nearly as we please by the equation $2Q = r\pi$, where r is a large enough integer.

ART. 17. *For Waves of Sufficient Length in the direction of flow, all Disturbances are Aperiodic, the values of p being given approximately by equation* (15).

The period-equation may be written in the form

$$1 + \frac{2a^2p'}{3\nu} + \frac{2a^4(21p'^2 + \beta^2l^2a^2)}{315\nu^2} + \frac{4a^6p'(9p'^2 + \beta^2l^2a^2)}{2835\nu^3}$$
$$+ \frac{2a^8(429p'^4 + 78p'^2\beta^2l^2a^2 + \beta^4l^4a^4)}{1216215\nu^4} + \frac{4a^{10}p'(117p'^4 + 30p'^2\beta^2l^2a^2 + \beta^4l^4a^4)}{18243225\nu^5} + \ldots = 0, \tag{27}$$

where $p' = p + \nu\lambda^2$, and accordingly if $\beta la^3/\nu$ is small enough, it is evident that no value of $(p + \nu\lambda^2)a^2/\nu$ is very small; hence, if la is large enough, all the values of

$$\{(p + \nu\lambda^2)a^2/\nu\}^3(\beta la^3/\nu)^{-2}, \quad \text{or} \quad (p + \nu\lambda^2)^3/(\nu l^2\beta^2)$$

can be as large as we please, and hence

$$\frac{2}{3}\left\{\frac{l\beta}{\nu}\left(-\frac{\nu\lambda^2 + p}{l\beta} \pm ai\right)^3\right\}^{\frac{1}{2}},$$

so large that the approximate forms of the I functions for large values of the parameter may be applied as accurately as we please, and it thus appears evident that, under such circumstances, *all* the values of p are given approximately by (15).

ART. 18. *A Rigorous Proof of last Proposition. Number of Roots in a Circular Contour of large Radius having Origin as Centre.*

A rigorous proof of the last statement presents some difficulties, however. Let p be any quantity, in general complex, not restricted to a value which satisfies the period-equation, and denote $p + \nu\lambda^2$ by p'; then, if la is sufficiently small

$$u_1 - u_2 = \frac{2}{3}\left\{\frac{l\beta}{\nu}\left(\frac{-p'}{l\beta} + ai\right)^3\right\}^{\frac{1}{2}} - \frac{2}{3}\left\{\frac{l\beta}{\nu}\left(\frac{-p'}{l\beta} - ai\right)^3\right\}^{\frac{1}{2}} \doteq 2ai\,(-p'/\nu)^{\frac{1}{2}}, \quad (27\text{A})$$

in the sense that the *difference between* the left- and the right-hand members can be made less than any assigned quantity by taking la small enough; for the difference may be made less than a certain multiple of $\beta la^2/(\nu p')^{\frac{1}{2}}$ as follows from the binomial theorem. If, under these circumstances, with the origin as centre, there is described a circle for which

$$\text{mod } 2a\,(-p'/\nu)^{\frac{1}{2}} = (r + \tfrac{1}{2})\pi, \quad (28)$$

r being zero, or any integer, it may be proved that the number of roots of the period-equation within this contour is r. (The circle might equally well be taken so that the right-hand member of (28) is any other quantity lying between $r\pi$ and $(r + 1)\pi$, and finitely different from both.) Let the equation be written in the form

$$u_1^{\frac{1}{3}}u_2^{\frac{1}{3}}[\{I_{-\frac{1}{3}}(u_1) - I_{\frac{1}{3}}(u_1)\}\{I_{-\frac{1}{3}}(u_2) + I_{\frac{1}{3}}(u_2)\} - \{I_{-\frac{1}{3}}(u_2) - I_{\frac{1}{3}}(u_2)\}\{I_{-\frac{1}{3}}(u_1) + I_{\frac{1}{3}}(u_1)\}] = 0. \quad (29)$$

A comparison with (8) shows that in this form the proper equation has been, for convenience, multiplied by $u_1^{\frac{1}{6}}u_2^{\frac{1}{6}}$.

With a view to examine the increase of argument of the left-hand member as p' describes the circumference of the circle, we first trace the changes in the approximate expression for it in the different portions of the region traversed.

In fig. 1, O denoting the origin, let A, A' on the axis of imaginary quantities denote the points βlai, $-\beta lai$; through A draw AL parallel

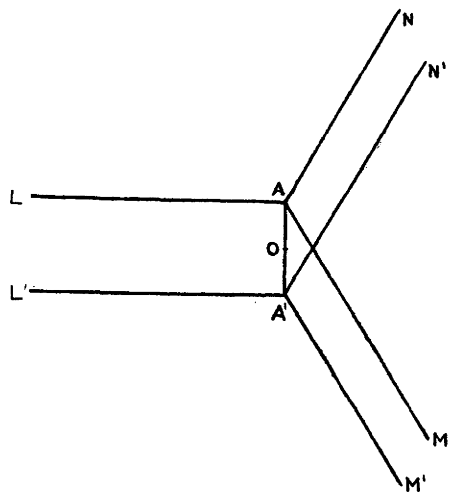

FIG. 1.

to the axis of real quantities and in the negative direction, and draw AM, AN making angles of $2\pi/3$ with AL; also draw $A'L'$, $A'M'$, $A'N'$ parallel to AL, AM, AN. Suppose p' starts from a point on the line AL; let the argument of each power of u_1 be zero in that position; and let the argument of each power of u_2 be zero when p' moves down to $A'L'$. When p' lies between AL, $A'L'$, since the ratio of its value, given by (28), to βla is large, the argument of u_1 is a small positive quantity, and that of u_2 a small negative quantity. Thus, in this region, from equations (9), (10), (11) we have

$$u_1^{\frac{1}{2}}(I_{-\frac{1}{3}}(u_1) - I_{\frac{1}{3}}(u_1)) \doteqdot (2/\pi)^{\frac{1}{2}} \sin \pi/3 \,.\, e^{-u_1}, \tag{30}$$

$$u_1^{\frac{1}{2}}(I_{-\frac{1}{3}}(u_1) + I_{\frac{1}{3}}(u_1)) \doteqdot (2/\pi)^{\frac{1}{2}} (e^{u_1} + i/2 \,.\, e^{-u_1}), \tag{31}$$

$$u_2^{\frac{1}{2}}(I_{-\frac{1}{3}}(u_2) - I_{\frac{1}{3}}(u_2)) \doteqdot (2/\pi)^{\frac{1}{2}} \sin \pi/3 \,.\, e^{-u_2}, \tag{32}$$

$$u_2^{\frac{1}{2}}(I_{-\frac{1}{3}}(u_2) + I_{\frac{1}{3}}(u_2)) \doteqdot (2/\pi)^{\frac{1}{2}} (e^{u_2} - i/2 \,.\, e^{-u_2}); \tag{33}$$

so that, omitting a constant factor, the left-hand member of (29) has the approximate form

$$e^{-u_1}(e^{u_2} - i/2 \,.\, e^{-u_2}) - e^{-u_2}(e^{u_1} + i/2 \,.\, e^{-u_1}), \tag{34}$$

or,

$$e^{u_2 - u_1} - e^{u_1 - u_2} - ie^{-u_1 - u_2}. \tag{35}$$

When p' crosses to the lower side of $A'L'$, since the argument of u_2 then becomes positive also, the factor $e^{u_2} - i/2 \, . \, e^{-u_2}$ of the right-hand member of (33) and of the first term of (34) is to be replaced by $e^{u_2} + i/2 \, . \, e^{-u_2}$, so that instead of (35), we have the simpler expression

$$e^{u_2 - u_1} - e^{u_1 - u_2}. \tag{36}$$

This expression remains valid, as p' travels round the circle until it passes into the region between AM, $A'M'$; here the argument of u_2 exceeds π; and it may be seen that the factor e^{-u_2} in (32) and in (36) is now replaced* by $e^{-u_2} + ie^{u_2}$, and that (36) now becomes

$$e^{u_2 - u_1} - e^{u_1 - u_2} - ie^{u_1 + u_2}. \tag{37}$$

When p' passes out of this region, the factor e^{-u_1} for a similar reason has to be replaced by $e^{-u_1} + ie^{u_1}$, and, accordingly, we now recover the simpler expression (36). This holds good again until p' passes into the space between the lines AN, $A'N'$; in so doing, the argument of u_2 is increased through 2π, and thus the factor e^{u_2} is changed into $e^{u_2} + ie^{-u_2}$, and (35) into

$$e^{u_2 - u_1} - e^{u_1 - u_2} + ie^{-u_1 - u_2}. \tag{38}$$

When p' crosses $A'N'$, the factor e^{u_1} is changed into $e^{u_1} + ie^{-u_1}$ from a similar cause, and we thus again recover the simple expression (36), which remains valid until p' reaches its starting-point on the line AL.

The final value of (36) is, however, not the same as the initial, but differs from it by a change of sign; for the initial and final values of u_1, and also those of u_2, are equal in magnitude and opposite in sign.

Again, under the circumstances stated, the simple expression (36) is in reality valid all round the contour; for the additional term in (35), (37), or (38), as the case may be, is small compared with the larger of the others. (It may be seen, however, that if the circumstances were such that the circular contour cut the productions of the lines AN, $A'M'$ between the lines AL, $A'L'$, it would not be legitimate in that region to omit the final term of (35); as will be shown below,† for sufficiently short waves there are

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* The law of discontinuity in the form of the approximate expressions for the Bessel functions was conveniently stated by Stokes ("On the Discontinuity of the Arbitrary Constants that appear as Multipliers of Semi-Convergent Series"; Acta Mathematica, xxvi., 1902; Collected Papers, v., p. 285). The substance of his statement is that of the two expressions—(1) e^u multiplied by a divergent series whose first term is unity, and (2) e^{-u} multiplied by a similar series—when the argument of u increases through an even multiple of π, (1) must be increased by $2i \cos r\pi$ times (2); and when through an odd multiple, (2) must be increased by $2i \cos r\pi$ times (1), in order that they may respectively continue to represent the same linear function of $x^{\frac{1}{2}} I_r(x)$ and $x^{\frac{1}{2}} I_{-r}(x)$. This may be seen, in fact, from equations (9), (10).

† Art. 21, p. 111.

complex roots for which p' lies near one or other of the productions mentioned.)

We have then to trace the change of argument of $e^{u_2-u_1} - e^{u_1-u_2}$ as p' describes this circular contour. It will be more convenient to suppose p' to start from, and stop at, the point of the circle midway between AL, $A'L'$. From (27A), (28) it is seen that, as p' describes the contour, the real part of $u_2 - u_1$ starts from an initial value zero, is continually positive, and ends with the value zero, while the imaginary part continually increases from

$$-(2r+1)\pi/2 \quad \text{to} \quad +(2r+1)\pi/2.$$

Thus, of the vectors $e^{u_2-u_1}$, $e^{u_1-u_2}$, the former is throughout the greater, except that their initial values are equal;* the former revolves in the positive direction, and the latter in the negative direction, each through an angle $(2r+1)\pi$; owing to the former being throughout the greater, the vector $e^{u_2-u_1} - e^{u_1-u_2}$, which is their difference, follows the direction of the former, oscillating about it, but never rotating round it,† making, indeed, always an acute angle with it. As the initial direction of this difference is the same as that of $e^{u_2-u_1}$, and as the same is true of the final directions, the total angle through which the vector difference rotates is the same as that through which $e^{u_2-u_1}$ rotates, i.e. a positive angle $(2r+1)\pi$. Thus, while p' describes the circle, the argument of the left-hand member of (29) increases by $(2r+1)\pi$. But the points A, A' are zeroes of the left-hand member of (29), extraneous to the proper period-equation; the increase in the argument of the extra factor $(u_1u_2)^{\frac{1}{2}}$, or in $(-p'+l\beta ai)^{\frac{1}{4}}(-p'-l\beta ai)^{\frac{1}{4}}$, is π. Subtracting this we obtain an increase of $2r\pi$ as that depending on the number of zeroes we wish to find; hence their number is r. But all the zeroes have been proved to lie between the lines AL, $A'L'$. By giving r the values $0, 1, 2$, etc., in succession, we see that there is no zero to the right of the arc of the first circle $r = 0$, and that there is one and only one zero in each of the quadrilateral spaces bounded by two consecutive circles and the parallel lines. And it has been already shown that in each such space there is one real zero given approximately by $u_1 - u_2 \doteqdot r\pi i$; hence, under the circumstances referred to at the beginning of the Art., this approximate equation gives all the zeroes.

And the same argument shows that whatever the value of la, if r is large enough, the number of zeroes lying inside the circle referred to in (28) is r.

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* But opposite, and the same statements hold, of course, for their final values.

† It is important to note that in the first and last quadrants of the circular contour the real part of $u_2 - u_1$ changes more rapidly (and in the first and last portions exceedingly more rapidly) than the imaginary part, so that when the vectors, which are represented only approximately by $e^{u_2-u_1}$ and $e^{u_1-u_2}$, are in the same direction, even for the first and the last times, the former is very much the greater.

ART. 19. *The Double Roots of the Period-Equation.*

As for waves of sufficient length in the direction of flow, all the values of p are real, it follows that, if this wave-length be supposed at first large and then to be gradually diminished, a value of p can become complex only by the wave-length passing through a value such that two real values of p become coincident.

Now, if we write

$$\left(\frac{l\beta}{\nu}\right)^{\frac{1}{3}}\left(ai - \frac{\nu\lambda^2 + p}{l\beta}\right) = Y_1, \quad \left(\frac{l\beta}{\nu}\right)^{\frac{1}{3}}\left(-ai - \frac{\nu\lambda^2 + p}{l\beta}\right) = Y_2, \qquad (39)$$

the period-equation in the notation of equations (6), (7) assumes the form

$$\phi(Y_1)\psi(Y_2) - \phi(Y_2)\psi(Y_1) = 0. \qquad (40)$$

If p has the real negative value which makes

$$Y_1^3 = Y_2^3 = \text{a real negative quantity,*}$$

the functions $\phi(Y_1)$, $\phi(Y_2)$ are identical; and the same is true of

$$Y_1^{-1}\psi(Y_1),\ Y_2^{-1}\psi(Y_2), \quad \text{and also of} \quad \psi'(Y_1),\ \psi'(Y_2);$$

accordingly, if this value of p just alluded to makes $\psi(Y_1)$, and therefore also $\psi(Y_2)$ vanish, this value of p is a double root of the period-equation. (If such a value of p, however, makes $\psi(Y_1)$, $\psi(Y_2)$ vanish instead, it is only a single root; for, to be a double root, it would require to make either $\psi'(Y)$ or $\phi(Y)$ vanish; but no root of $J_n(x) = 0$ can satisfy either $J'_n(x) = 0$ or $J_{-n}(x) = 0$.) Thus, there are double roots p for certain values of l, p and l being given by the equations

$$\nu\lambda^2 + p = -3^{-\frac{1}{2}}l\beta a, \quad J_{\frac{1}{3}}\left\{\frac{2}{3}\left(\frac{8l\beta a^3}{3\sqrt{3}\nu}\right)^{\frac{1}{2}}\right\} = 0. \qquad (41)$$

It may be proved, also, that these equations give the only double roots. The equation

$$d/dp\,\{\phi(Y_1)\psi(Y_2) - \phi(Y_2)\psi(Y_1)\} = 0, \qquad (42)$$

which a double root must satisfy, when combined with (40), gives

$$\{\phi(Y_1)\}^2\{\phi(Y_2)\psi'(Y_2) - \phi'(Y_2)\psi(Y_2)\} = \{\phi(Y_2)\}^2\{\phi(Y_1)\psi'(Y_1) - \phi'(Y_1)\psi(Y_1)\}. \qquad (43)$$

But, from the linear differential equation satisfied by ϕ, ψ, we have, for all values of the parameter,

$$\phi(Y)\psi'(Y) - \phi'(Y)\psi(Y) = \text{constant};$$

so that (43) is equivalent to

$$\{\phi(Y_1)\}^2 = \{\phi(Y_2)\}^2; \qquad (44)$$

* For any such value p' is represented by the point C (fig. 2, p. 108).

and thus the equations to be satisfied in case of a double root are either

$$\phi(Y_1) = \phi(Y_2), \quad \text{and} \quad \psi(Y_1) = \psi(Y_2), \tag{45}$$

or else

$$\phi(Y_1) = -\phi(Y_2), \quad \text{and} \quad \psi(Y_1) = -\psi(Y_2). \tag{46}$$

The former alternative is equivalent to the statement that $\phi(Y_1), \psi(Y_1)$ should both be purely real; the latter, that they should both be purely imaginary. In either case, there would exist some equation of the type

$$\phi(Y_1) + C\psi(Y_1) = 0, \tag{47}$$

in which C is some real quantity, except either ϕ or ψ vanishes (for both Y_1 and Y_2). Of the two exceptional cases, that in which

$$\psi(Y_1) = \psi(Y_2) = 0, \quad \phi(Y_1) = \phi(Y_2), \tag{48}$$

is the one already referred to; for, as a Bessel function* can vanish only for real values of the argument, the former pair of these equations requires Y_1^3 and Y_2^3 to be real, negative, and therefore, by (39), equal, quantities. The second exceptional case, i.e.

$$\phi(Y_1) = \phi(Y_2) = 0, \quad \psi(Y_1) = \psi(Y_2) \tag{49}$$

is impossible, for the former pair of equations again requires that Y_1^3 and Y_2^3 should be real negative equal quantities. Then, since Y_1 cannot be equal to Y_2, the second pair would imply that $\psi(Y_1)$ and $\psi(Y_2)$ should both vanish; this would recover the former exceptional case, though it is impossible that ϕ, ψ should vanish together. Thus we are driven back to equation (47). But this cannot be satisfied by a complex value of Y^3. We may rest this last statement on the general theorem that, if n lies between ± 1, any expression of the form

$$I_{-n}(x) + CI_n(x),$$

where C is a real quantity, and every power of x has its principal value, can vanish for, at most, only one value of x, and this a real positive one.† Or it may be established independently as follows: Denote by $\chi(Y)$ the left-hand member of (47) with Y_1 replaced by Y;‡ and suppose, if possible, it vanishes for Y_1 and Y_2, complementary complex values; we evidently have

$$d^2\chi(aY_1)/da^2 = aY_1^3\chi(aY_1),$$

$$d^2\chi(aY_2)/da^2 = aY_2^3\chi(aY_2);$$

from which we deduce

$$\chi(aY_1)d^2\chi(aY_2)/da^2 - \chi(aY_2)d^2\chi(aY_1)/da^2 = (Y_2^3 - Y_1^3)a\chi(aY_1)\chi(aY_2);$$

* Of order greater than -1, as here.

† Unless $n = \frac{1}{2}$, in which case it may be a negative one.

‡ By Y is denoted $(l\beta/\nu)^{\frac{1}{3}}(-\nu\lambda^2 - p - l\beta yi)/l\beta$ as in (6).

on multiplying by $d\alpha$, and integrating between the limits 0 and 1, we obtain

$$\chi(Y_1)\chi'(Y_2) - \chi(Y_2)\chi'(Y_1) = (Y_2^3 - Y_1^3)\int_0^1 \alpha\chi(\alpha Y_1)\chi(\alpha Y_2)\,d\alpha; \qquad (49\text{ A})$$

by supposition the left-hand member is zero, while the integrand on the right, being the product of conjugate complex factors, is essentially positive; accordingly Y_1^3 and Y_2^3 must be equal; and, on substituting in succession Y_1, Y_2 in (47), we evidently return to the special exceptional cases again.

ART. 20. *The March of the Roots, as the Wave-Length, in Direction of Flow decreases. A finite Number of Disturbances become Oscillatory.*

In fig. 2, let O be the origin, A, A' the points βlai, $-\beta lai$, and C the point

$$-\beta la/\sqrt{3}.$$

As proved in Art. 19, when a double root occurs, the value of p' is represented by the point C.

I desire to make use of some expression for the error in terminating, after an assigned term, the divergent series which occur in connexion with the Bessel functions; a partial statement as to this error has been made in

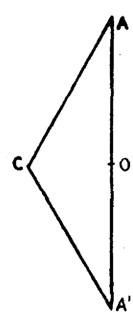

FIG. 2.

connexion with equation (9); it may now be completed by stating that, in that equation, if the argument of x is $\pm(\pi - \gamma)$, γ being acute, one form of the multiplier there alluded to is

$$\operatorname{cosec}(\theta + \gamma)(\sec\theta)^{\frac{1}{2}+k+s},$$

where θ is any acute angle such that $\theta + \gamma$ is also acute; in the case in hand we may conveniently take θ to be zero, and use the theorem that the error is less than the next term multiplied by $\operatorname{cosec}\gamma$. And as when $k = \frac{1}{3}$, $\frac{1}{2} - k + s$ is positive, even when s is zero, we may use this form of remainder after any number (even zero) of terms. When p' lies between C and O, the argument of u_1 lies between $\pi/2$ and $3\pi/4$, and that of u_2 between $-\pi/2$ and $-3\pi/4$, so that, when the period-equation is written in the form (23), we may take in the notation of (26), $A = 5/72$, $B = 5\sqrt{2}/72$. We shall not be using the approximations in any case in which the value of $|u_1|$ or $|u_2|$ at C is less than $3\pi/4$; consequently, at any point between C and O, the value of $|u|$ exceeds

$$(\sqrt{3}/2)^{\frac{3}{2}} \,.\, 3\pi/4 \quad \text{or} \quad 1{\cdot}8989,$$

and thus the fractional error in e^{u_1} or e^{u_2} is less than $1/27$, and that in e^{-u_1} or e^{-u_2} less than $\sqrt{2}/27$. Thus, if the period-equation be brought to the form

$$-i(e^{2u_1} - e^{2u_2}) + 1 \doteqdot 0 \tag{50}$$

by dividing across by the factor which will make the third term rigorously accurate, the fractional error in e^{2u_1} or e^{2u_2} is less than

$$\left(1 + \frac{1}{27}\right) \bigg/ \left(1 - \frac{\sqrt{2}}{27}\right) - 1,$$

and therefore less than $1/10$. Thus the correct left-hand member lies between

$$e^{2P}(2 \sin 2Q \pm 1/5) + 1.$$

Let us suppose that at C, $u_1 = u_2 = n\pi i + \pi i/4$, where n is unity, or any higher integer. At C the left-hand member lies between the limits

$$2 \sin \pi/2 \pm 1/5 + 1,$$

and is therefore positive. As p' travels from C towards O, the factor $2 \sin 2Q \pm 1/5$ remains positive, certainly until $2Q$ decreases by $\pi/3$, at which stage $2P$ has decreased algebraically by more than $\pi/3$, (for it may easily be seen by differentiating $(-p' + ai)^{\frac{3}{2}}$ that its real part decreases algebraically as p' moves towards O at a rate which, measured absolutely, is greater than the rate of decrease of its imaginary part), and hence $e^{2P} < e^{-\pi/3} < e^{-1}$; everywhere between this point and O, $e^{2P}(2 \sin 2Q \pm 1/5)$ is numerically less than $(2\frac{1}{5})\,e^{-1}$, and thus the left-hand member is positive. Under these circumstances, then, there is no root of the period-equation for which p' lies between C and O.

Let us next suppose that, at C, $u_1 = iQ = n\pi i - \pi i/4$, n being unity or any higher integer. At C the left-hand member of (50) lies between the limits

$$-2 \sin \pi/2 \pm 1/5 + 1,$$

and is therefore negative. Again, at O the left-hand member lies between the limits

$$e^{2P}(2 \sin 2Q \pm 1/5) + 1,$$

where P is negative and numerically greater than $(1{\cdot}9)/\sqrt{2}$, this being its value in the case $n = 1$; from this it is clear that the left-hand member is essentially positive. Thus, under these circumstances, there must be some odd number of roots for which p' lies between C and O.

Now, the roots of the equations $J_{-\frac{1}{3}}(x) = 0$, $J_{\frac{1}{3}}(x) = 0$, occur alternately; those of the former are approximately $x = r\pi + 7\pi/12$, and those of the latter $x = r\pi + 11\pi/12$, where r is zero, or any positive integer; and, as has been proved in Art. 19, whenever the value of p' at C is such that the corresponding value of $u_1 i$ (or $u_2 i$) is a zero of $J_{\frac{1}{3}}(x)$, this value of p' is a double root of the period-equation. Hence we can trace the effect of diminishing the wave-length in the direction of flow on the nature of the roots of the period-equation. Starting with a very small value of la, if we gradually increase it until

$$\frac{2}{3}\left\{\frac{l\beta}{\nu}\left(\frac{2}{\sqrt{3}}\,a\right)^3\right\}^{\frac{1}{2}} \quad \text{or} \quad (32l\beta a^3/(27\sqrt{3}\,.\,\nu))^{\frac{1}{2}} \tag{51}$$

becomes equal in value to the lowest zero of $J_{-\frac{1}{3}}(x)$, the smallest value of p' is represented by the point C; if we further increase l, this value passes between C and O, and so remains until the expression (51) becomes equal to the lowest zero of $J_{\frac{1}{3}}(x)$; at this stage two roots of the period-equation coincide at C. On increasing the la still further, these two roots become complex, and there is now no root between C and O until the expression (51) becomes equal to the next zero of $J_{-\frac{1}{3}}(x)$, at which stage a root passes C, to return to it, and, coalescing with another, become a double one when (51) becomes equal to the second zero of $J_{\frac{1}{3}}(x)$; after this these two become complex and different; and so on.

That a pair of roots do, indeed, become imaginary as la increases through the value which makes them coincident, may be seen as follows:—It has been shown that when la is sufficiently small, there is one, and only one, root between the real values for which

$$u_1 - u_2 = (2r \pm 1)\pi i/2\,; \tag{52}$$

now, the roots are continuous functions of a, i.e. dp'/da is finite (except when p' is a double root); hence, the only manner in which this distribution of roots could be altered would be by a root passing through a point given by (52). But, by making use of the above expressions for the limits of error, it is easy to prove that this is impossible; thus, two real roots do disappear—one from the left and one from the right of C—while the value of u_1 at C changes from $(r - \frac{1}{4})\pi i$ to $(r + \frac{1}{4})\pi i$. But, from the statement in the final sentence of Art. 18, p. 105, these roots continue to exist, and must therefore be complex.

Thus, the greatest wave-length in the direction of flow for which a disturbance can be oscillatory is $2\pi/l$, where

$$\{32\beta l a^3/(27\sqrt{3}\nu)\}^{\frac{1}{2}} = \text{ the lowest zero of } J_{\frac{1}{3}}(x) \doteqdot 2{\cdot}87. \tag{53}$$

ART. 21. *The Approximate Values of the Complex Roots.*

If the point p' lies to the right of the line $A'C$ (fig. 2), the argument of u_2 lies between $-\pi/2$ and $-3\pi/4$, so that if u_2 is large enough, e^{u_2} is small compared with e^{-u_2}; thus, the period-equation takes the approximate form

$$-ie^{u_1} + e^{-u_1} \doteqdot 0, \tag{54}$$

giving

$$u_1 \doteqdot (r\pi + 3\pi/4)i, \tag{55}$$

where r is zero or any positive integer. This assigns to p' a position P such that

$$\tfrac{2}{3}(PB^3/\nu)^{\frac{3}{2}}/l\beta \doteqdot (r\pi + 3\pi/4)\,i, \tag{56}$$

giving

$$p = p' - \nu\lambda^2 \doteqdot -\nu\lambda^2 - \frac{1}{2}\left(\frac{3}{2}\cdot\frac{4r+3}{4}\pi\right)^{\frac{2}{3}}\left(\nu l^2\beta^2\right)^{\frac{1}{3}}$$

$$+\, i\left\{\beta la - \frac{\sqrt{3}}{2}\left(\frac{3}{2}\cdot\frac{4r+3}{4}\pi\right)^{\frac{2}{3}}\left(\nu l^2\beta^2\right)^{\frac{1}{3}}\right\}, \tag{57}$$

r being any positive integer (including zero), provided r is not so great as to make the coefficient of i negative; (in that case, we return to the real roots).

A more correct, though still only approximate, equation is that which makes the numerical value of u_1 satisfy

$$J_{\frac{1}{3}}|u| + J_{-\frac{1}{3}}|u| = 0. \tag{58}$$

Equation (58), or its approximate form (56), becomes less and less accurate if the position it assigns for p' is near C; as we have seen, p' coincides with C for values of u_1 satisfying the equation

$$J_{\frac{1}{3}}(u_1 i) = 0, \quad \text{or} \quad u_1 \doteqdot (r\pi + 11\pi/12)i;$$

the $r + 3/4$ of (55) being thus replaced by $r + 11/12$.

It is seen that these values of p' all lie close to the line CA; but it may be seen that the correct values cannot actually lie on the line except when at C. And as the roots we have so found, taken along with their images in the axis of real quantities, just equal in number those which have been proved to be complex, all the roots have been accounted for and approximately ascertained.

ART. 22. *In the most Persistent Disturbance, v is a Function of y only.*

When the wave-lengths in the directions of x and z are increased indefinitely, i.e., when the velocity-component v is made a function of y only, λ and l are both zero, and the values of p are given by $p = \nu r^2\pi^2/4a^2$, r being any integer, as may be seen from (15), or, by returning to (1), and the lowest numerical value is that for which r is unity. For any finite value of l, the value of the real part of p', or $p + \nu l^2 + \nu n^2$, and therefore, *a fortiori*, that of p, is numerically greater than in this case. This may be proved as follows.

Considering, firstly, the real values of p', if we write, as in Art. 15, $S = P + iQ$, and integrate equation (20) from $-a$ to y, we obtain

$$\nu\{PdQ/dy - QdP/dy\} = l\beta \int_{-a}^{y} y\,(P^2 + Q^2)\,dy. \tag{59}$$

Since $P^2 + Q^2$ is not changed by changing the sign of y, the right-hand member is essentially of opposite sign to l between $\pm a$, except that it is zero at $\pm a$; consequently so is the left-hand member. Hence we may infer that between every two real zeros of P, provided $y = \pm a$ be not one of them, there lies one zero of Q, and between every two of Q, with the same exception, there lies one of P. From the forms assumed by (16), (17), when p is real, evidently of the two functions P, Q one is odd, the other even; we will choose P even, Q odd. Then Q vanishes when y is zero; it seems to be the case that for given values of l, n, in the disturbance which has the smallest numerical value of p, with this exception, neither P nor Q can vanish for any other values than $\pm a$; if, however, this be not the case, we have just proved that as y increases from zero it will reach a zero of P before another of Q; and thus in any event a zero of P not later than another of Q. When y is zero it results from (59) that if P be taken positive as it may, dQ/dy is of sign opposite to that of l, and thus as y increases from zero, Q also has its sign opposite to that of l. Consequently in the equation which (16) now becomes, viz.:

$$\nu d^2P/dy^2 = p'P - \beta lyQ, \tag{60}$$

the first term on the right is negative, and the second positive. Thus the variation of P, until it becomes zero, is analogous to that of the displacement of a particle ν subject to a force to a fixed point, which force is less than the displacement multiplied by $-p'$; and the particle starts from rest. The time which elapses until the particle reaches the centre is greater than

$$\frac{\pi}{2}\left(\frac{-\nu}{p'}\right)^{\frac{1}{2}}.$$

Therefore, in the problem which is the subject of discussion, the value of y for which P first vanishes—a value which, as we have seen, cannot exceed a—is greater than

$$\frac{\pi}{2}\left(\frac{-\nu}{p'}\right)^{\frac{1}{2}}, \text{ i.e., } \ -p' > \nu\pi^2/4a^2.$$

Thus the result is established for real values of p'.

I have not succeeded in obtaining a rigorous proof for complex values of p. Whenever such roots occur, the approximate value, however, of the real part of the first complex value of p', as given by (57), is much greater than $\nu\pi^2/4a^2$. In fact, if a be regarded as fixed, and l is increased from zero, when the first root of the period-equation reaches C, u_1 being then the lowest root of the equation

$$J_{-\frac{1}{3}}\{32\beta la^3/(27\sqrt{3}\,\nu)\}^{\frac{1}{2}} = 0,$$

(which is a little greater than $7\pi/12$), the numerical value of p' is slightly greater than $(147/128)\,(\nu\pi^2/4a^2)$. No complex root occurs, however, until l is further increased to such a value that

$$J_{\frac{1}{3}}\{32\beta la^3/(27\sqrt{3}\nu)\}^{\frac{1}{2}} = 0,$$

as the lowest value for which $J_{\frac{1}{3}}(x)$ vanishes slightly exceeds $11\pi/12$, the corresponding value of p' is a little greater than $(363/128)(\nu\pi^2/4a^2)$. And, in the approximate formula (57) for the complex roots, l, and therefore also $\nu l^2\beta^2$, has a larger value than in this critical case, while the coefficient of $(\nu l^2\beta^2)^{\frac{1}{3}}$ in the real portion is decreased in the ratio $(9/11)^{\frac{2}{3}}$; the approximate value of the real part of p' is thus numerically greater than

$$\frac{363}{128}\cdot\left(\frac{9}{11}\right)^{\frac{2}{3}}\cdot\frac{\nu\pi^2}{4a^2}.$$

It does not seem possible that this approximate value could be so far wrong that the actual value should be so small as $\nu\pi^2/4a^2$.

For small values of la a further approximation to the r^{th} root of the period equation is given by

$$(-p'a/\nu)^{\frac{1}{2}} \doteqdot \frac{r\pi}{2}\left\{1 - \left(\frac{2}{3r^4\pi^4} - \frac{10}{r^6\pi^6}\right)\frac{\beta^2 l^2 a^6}{\nu^2}\right\}. \tag{61}$$

It thus seems probable that, as la is gradually increased from zero, the lowest value of $-p'$ continually increases, and the other values of $-p'$ (but not necessarily those of $-p$) continually decrease until they become complex.

ART. 23. *Equations for resolving an Arbitrary Disturbance into the Fundamental ones: Inability to use them.*

The problem of resolving any arbitrary disturbance (subject to the boundary-conditions $\nabla^2 v = 0$) evidently reduces to that of expressing an arbitrary function of y which vanishes when $y = \pm\, a$, in terms of the functions S which correspond to the free modes of disturbance already investigated, having the values of l, λ assigned. If S_1, S_2, be functions corresponding to two different possible values p_1, p_2 of p, from the equations

$$\nu d^2S_1/dy^2 = (\nu\lambda^2 + p_1 + il\beta y) S_1,$$

$$\nu d^2S_2/dy^2 = (\nu\lambda^2 + p_2 + il\beta y) S_2,$$

there results

$$\nu(S_1 d^2S_2/dy^2 - S_2 d^2S_1/dy^2) = (p_2 - p_1) S_1 S_2,$$

and by integration between the limits $\pm a$,

$$(p_2 - p_1)\int_{-a}^{a} S_1 S_2 dy = \nu \left| S_1 dS_2/dy - S_2 dS_1/dy \right|_{-a}^{a}. \tag{62}$$

If p_2 and p_1 are different values for which S_1, S_2 vanish at the limits, this gives

$$\int_{-a}^{a} S_1 S_2 dy = 0. \tag{63}$$

If, in the formula (62), we write $p_2 = p_1 + \delta p_1$, divide by δp_1, and then suppose δp_1 to diminish indefinitely, we obtain

$$\int_{-a}^{a} S_1^2 dy = \nu \left| S_1 \frac{d^2S_1}{dy dp_1} - \frac{dS_1}{dy}\frac{dS_1}{dp_1} \right|_{-a}^{a}$$

$$= -\nu \left| \frac{dS_1}{dy}\frac{dS_1}{dp_1} \right|_{-a}^{a} \tag{64}$$

since S_1 vanishes at both limits.

Thus, if we *assume* the possibility of expanding an arbitrary function, $f(y)$, in a series of the form

$$\sum_1^\infty A_r S_r(y),$$

the coefficients are from (63), (64) determined by equations of the form

$$-\nu A_r \left| \frac{dS_r}{dy}\frac{dS_r}{dp_r} \right|_{-a}^{a} = \int_{-a}^{a} f(y) S_r(y) dy. \tag{65}$$

Should the period-equation have a double root p, in which case that portion of the complete disturbance which involves e^{pt} takes the form

$$ASe^{pt} + B(e^{pt}dS/dp + tSe^{pt}),$$

the expansion of $f(y)$, the value of S at the time $t = 0$, has to include a term BdS/dp as well as AS, and (65) fails to determine A, B. The investigation necessary to find their values is somewhat longer, and it appears unnecessary to give it.

I have not succeeded in applying these formulæ to any initial disturbance of the simplest type, such as that discussed by Lord Kelvin. Towards so doing, the evaluation, accurate or approximate, of the coefficients A by means of (65) would be only one step. Were this accomplished, we would have

$$S = \sum_{1}^{\infty} A_r S_r(y) e^{p_r t}, \tag{65 A}$$

and V would have to be found from this, by the aid of (2), and found in a form suitable for arithmetical comparisons.

It may be noted that although, from the results of Chap. I., above, and those of Part I., there is good reason to suppose that, for a suitably chosen initial disturbance, V may increase very much, this is not the case with S. On the contrary, it readily follows from (2) of Chap I. that the average value of S^2 throughout the liquid diminishes continuously and indefinitely; a similar contrast between decreasing S and increasing V may be noted for the disturbances discussed in Chap. I., Arts. 2 and 10–12.

ART. 24. *The Case of Boundary-Conditions* $dS/dy = 0$.

If the assigned boundary-conditions are that dS/dy should vanish at each of the boundary-planes, the period-equation is obtained by making, in the notation of equations (5), (6), (7),

$$A'\psi'(Y) + B'\phi'(Y)$$

vanish at the boundaries; but

$$\psi'(Y) = 3^{-\frac{2}{3}}\Pi(-\tfrac{2}{3})YI_{-\frac{2}{3}}(\tfrac{2}{3}Y^{\frac{3}{2}}),$$

$$2\phi'(Y) = 3^{\frac{2}{3}}\Pi(\tfrac{2}{3})YI_{\frac{2}{3}}(\tfrac{2}{3}Y^{\frac{3}{2}});$$

so that the equation is similar to (8), except that the I functions are of order $\pm\frac{2}{3}$.

For large values of p' whose real part is negative, the approximate form of this equation is

$$e^{u_1-u_2} - e^{u_2-u_1} - ie^{-u_1-u_2} \doteqdot 0. \tag{66}$$

Obviously it may be proved, as in Art. 16, that for all values of l, n, there are an infinite number of aperiodic disturbances, the values of p being given approximately by (14), (15) again.

Evidently, too, if la is small enough, in (15) r may be taken to be *any* integer, even unity.

But an investigation almost identical with that of Art. 18 proves that, for all integral values of r (including zero), if la be small enough, and for all values of la, if r be large enough, the number of roots inside the circular contour for which

$$\text{mod } 2a\,(-p'/\nu)^{\frac{1}{2}} = (r+\tfrac{1}{2})\pi$$

is $r+1$, *one more than with the boundary-conditions* $S = 0$. This difference in number is due to the fact that (66) has to be multiplied, instead of divided as is the case with (23), by

$$(-p' + l\beta ai)^{\frac{1}{4}}(-p' - l\beta ai)^{\frac{1}{4}},$$

in order that it may represent, for large values of p', the true period-equation.

Accordingly, when la is very small, the period-equation has *one root not given by* (15). This root gives a value to p' which is itself very small and diminishes indefinitely with la. In fact, if la is zero, one value of p' is zero; this may be seen by noting that when la is zero, $Y_1 = Y_2$, in the notation of (39); p' will now be zero if $Y_1 = Y_2 = 0$; and it is evident that these values satisfy the period-equation, *after* its division by $Y_1 - Y_2$, or an equivalent differentiation, which is a necessary preliminary. If, returning to (1), in it we replace l by zero, we do indeed obtain a root, p' = zero, corresponding to a disturbance in which S is constant, in time and in space.

Thus, if la be small enough, here again all the disturbances are aperiodic, and all the roots are accounted for by (15), with the exception of this one, which we may regard as also included in (15) on making r zero.

It is readily seen that a value of p' occurs at C (fig. 2, p. 108), whenever at this point

$$I_{-\frac{2}{3}}(u) = 0, \text{ i.e. } u \doteqdot (r\pi + 5\pi/12)i,$$

or

$$I_{\frac{2}{3}}(u) = 0, \text{ i.e. } u \doteqdot (r\pi + 13\pi/12)i,$$

r being zero or any positive integer. The former set are double roots; and it may be proved much as in Art. 19 that these are the only double roots.

We may trace, as in Art. 20, the effect of diminishing the wave-length in the direction of flow on the nature of the roots. When la is exceedingly small, one value of p' is close to O (fig. 2), and all the others to the left of C; as l is gradually increased, all the roots move towards C until the expression (51) becomes equal to the lowest zero of $J_{-\frac{2}{3}}(x)$; at this stage two values of p' coincide at C. On increasing l still further, these two roots become complex, and there is now no value between C and O until (51) becomes equal to the lowest zero of $J_{\frac{2}{3}}(x)$ when a value of p' passes C, to return to it and in coincidence with another become a double root when (51) becomes equal to the next zero of $J_{-\frac{2}{3}}(x)$; after this these two become complex and different; and so on.

The greatest wave-length in the direction of flow for which a disturbance can be oscillatory is thus $2\pi/l$, where

$$\{32\beta la^3/(27\sqrt{3}\nu)\}^{\frac{1}{2}} = \text{the lowest zero of } J_{-\frac{2}{3}}(x) \doteqdot 1{\cdot}2. \qquad (67)$$

There are a finite number of complex roots, those whose imaginary parts are positive being given, when not too near C, by the approximate equation

$$e^{u_1} - ie^{-u_1} \doteqdot 0,$$

or,

$$u_1 \doteqdot r\pi + \pi/4, \qquad (68)$$

where r is zero or any positive integer; and, more accurately, by

$$J_{-\frac{2}{3}}\,|u_1| - J_{\frac{2}{3}}|u_1| \doteqdot 0;$$

the second term of (66) is now small compared with the other two. These complex values of p', of course, as before, lie close to the line CA, and their conjugates close to CA'.

It is seen that here again all the roots which exist have been accounted for and approximately located.

It will be noticed that, approximately, when la is large, the real roots, if not too near C, are the same as when the boundary-conditions are $S = 0$; the complex roots are different, however; this is the only evidence I have noticed against the view that, for disturbances whose wave-lengths in all directions are small, the question of stability is little affected by the precise boundary-conditions.

ART. 25. *The Case of Boundary-Conditions $V = 0$, $dV/dy = 0$: Failure to obtain any Simple Proof that fundamental Disturbances are Stable.*

With the boundary-conditions $V = 0$, $dV/dy = 0$, I am unable to give any simple proof by any method analogous to that of Art. 15 that the fundamental modes of disturbance are exponentially stable. We obtain, however, the same limits for the imaginary parts of the values of p, viz., $\pm l\beta ai$. The equation satisfied by V being

$$[d^2/dy^2 - \{\lambda^2 + (p + il\beta y)/\nu\}](d^2/dy^2 - \lambda^2)\,V = 0,$$

if we write $V = V_1 + iV_2$, $p = \theta + i\phi$, separate the real and the imaginary parts, multiply one equation by V_1, the other by V_2, add, and integrate between the limits $\pm a$, we readily obtain

$$\int_{-a}^{a} (\phi + l\beta y)[(dV_1/dy)^2 + (dV_2/dy)^2 + \lambda^2(V_1^2 + V_2^2)]dy = 0, \qquad (69)$$

from which it follows that $\phi + l\beta y$ must change sign between the limits of y.

I have also been unable to obtain any equations analogous to (63), (64) Art. 23, by the aid of which any arbitrary free disturbance may be resolved into its constituent fundamental ones.

ART. 26. *Derivation of the Period-Equation: Its approximate Form.*

The solution of (1) being denoted by S, V may be expressed in the form

$$V = \frac{1}{2\lambda}\left\{e^{\lambda y}\int Se^{-\lambda y}\,dy - e^{-\lambda y}\int Se^{\lambda y}\,dy\right\},$$

whence

$$dV/dy = \tfrac{1}{2}\left\{e^{\lambda y}\int Se^{-\lambda y}dy + e^{-\lambda y}\int Se^{\lambda y}dy\right\}.$$

The boundary-conditions thus lead to the period-equation

$$\int_{-a}^{a} S_1 e^{\lambda y} dy \int_{-a}^{a} S_2 e^{-\lambda y} dy - \int_{-a}^{a} S_1 e^{-\lambda y} dy \int_{-a}^{a} S_2 e^{\lambda y} dy = 0, \qquad (70)$$

where S_1, S_2 are any two independent solutions of (1).

A laborious development of this equation in ascending powers of p' threw little light on the nature of the roots; every term in the equation appears to have the same sign, however.

On the supposition, justified to some extent by results, that for all the roots the quantities which occur as variables in the Bessel functions in S are large, an equation approximately equivalent to this may be obtained. As approximate forms of S are $(-p' - l\beta y i)^{-\frac{1}{4}} . e^{\pm u}$, where

$$u = \frac{2}{3}\left(\frac{l\beta}{\nu}\right)^{\frac{1}{2}} \left(\frac{-p' - l\beta y i}{l\beta}\right)^{\frac{3}{2}}, \qquad (71)$$

it might appear that we would be justified in using these exponential forms in the integrands, and replacing, for example,

$$\int_{-a}^{a} (-p' - l\beta y i)^{-\frac{1}{4}} e^{u+\lambda y} dy$$

by

$$\Big| (-p' - l\beta y i)^{-\frac{1}{4}} e^{u+\lambda y} / (\lambda + du/dy) \Big|_{-a}^{a}$$

Irrespective of the delicate considerations of the discontinuity in the forms of the approximate expressions for the Bessel functions, this procedure would not, however, be *prima facie* justifiable unless it were possible, regarding iy as a complex quantity, to connect the limits of integration by a path along which the real part of $u + \lambda y$ continuously increased, or continuously decreased, which is not always possible. I therefore considered more fully the functions $\int e^{\pm\lambda y} S dy$; but the approximate form finally obtained for the period-equation proved so intractable that it does not appear justifiable to go into details. In the region in which the roots appear to actually lie, viz., one in which p' has its real part negative, and its imaginary part between the limits $\pm\, l\beta a i$, the form is

$$\Bigg\{(-p'/(l\beta) + ai)^{-\frac{1}{4}} \left[\frac{Exp\,(\lambda a + u_1)}{\lambda + i((-p' + l\beta a i)/\nu)^{\frac{1}{2}}} + \frac{i\,Exp\,(\lambda a - u_1)}{\lambda - i((-p' + l\beta a i)/\nu)^{\frac{1}{2}}}\right]$$

$$+ 2\pi^{\frac{1}{2}} \left(\frac{\nu}{l\beta}\right)^{\frac{1}{4}} Exp\left(\frac{\nu\lambda^3 i - 3\lambda p' i}{3 l\beta}\right) - (-p'/(l\beta) - ai)^{-\frac{1}{4}} \frac{Exp\,(-\lambda a + u_2)}{\lambda + i((-p' - l\beta a i)/\nu)^{\frac{1}{2}}}\Bigg\}$$

$$\times \left\{\frac{(-p'/(l\beta) + ai)^{-\frac{1}{4}}}{-\lambda - i((-p' + l\beta a i)/\nu)^{\frac{1}{2}}} Exp(-\lambda a - u_1) - \frac{(-p'/(l\beta) - ai)^{-\frac{1}{4}}}{-\lambda - i((-p' - l\beta a i)/\nu)^{\frac{1}{2}}} Exp\,(\lambda a - u_2)\right\}$$

$$
\begin{aligned}
&-\Bigg\{(-p'/(l\beta)+ai)^{-\frac{1}{4}}\left[\frac{Exp\,(-\lambda a+u_1)}{-\lambda+i((-p'+l\beta ai)/\nu)^{\frac{1}{2}}}+\frac{i\,Exp\,(-\lambda a-u_1)}{-\lambda-i((-p'+l\beta ai)/\nu)^{\frac{1}{2}}}\right] \\
&+2\pi^{\frac{1}{2}}\left(\frac{\nu}{l\beta}\right)^{\frac{1}{4}}Exp\left(\frac{-\nu\lambda^3 i+3\lambda p' i}{3l\beta}\right)-(-p'/(l\beta)-ai)^{-\frac{1}{4}}\,\frac{Exp\,(\lambda a+u_2)}{(-\lambda+i((-p'-l\beta ai)/\nu)^{\frac{1}{2}}}\Bigg\} \\
&\times\left\{\frac{(-p'/(l\beta)+ai)^{-\frac{1}{4}}}{\lambda-i((-p'+l\beta ai)/\nu)^{\frac{1}{2}}}\,Exp\,(\lambda a-u_1)-\frac{(-p'/(l\beta)-ai)^{-\frac{1}{4}}}{\lambda-i((-p'-l\beta ai)/\nu)^{\frac{1}{2}}}\,Exp\,(-\lambda a-u_2)\right\}\doteq 0,
\end{aligned}
\tag{72}
$$

u_1 denoting, as before, $\frac{2}{3}(l\beta/\nu)^{\frac{1}{2}}\{-p'(l\beta)+ai\}^{\frac{3}{2}}$, and u_2 the corresponding expression with the sign of a changed.

ART. 27. *Some Results.*

It appears that the period-equation has no roots for which the real part of p (or even that of p') is positive. If the real part of p is supposed positive, the equation assumes a simpler form; the first expression within the $\{\quad\}$ is to be replaced by

$$
(-p'/(l\beta)+ai)^{-\frac{1}{4}}\frac{Exp\,(\lambda a+u_1)}{\lambda+i((-p'+l\beta ai)/\nu)^{\frac{1}{2}}}-(-p'/(l\beta)-ai)^{-\frac{1}{4}}\frac{Exp\,(-\lambda a+u_2)}{\lambda+i\,((-p'-l\beta ai)/\nu)^{\frac{1}{2}}};
\tag{73}
$$

and the third is to be similarly replaced by the first and last of the four terms which constitute it. In fact, if the real part of p (though not necessarily if merely that of p') is positive, that of any one of the expressions $\pm\lambda y \pm u$ either continually increases or continually decreases as y changes from $-a$ to $+a$; and accordingly it seems evident that we may proceed as indicated in the third paragraph of the preceding Article, and thus obtain this modified form of the period-equation. If we now consider the terms in the equation which are most important, it will be found that it is necessary that $e^{4\lambda a}$ should be complex or less than unity, which is, of course, impossible.

In using these approximate forms there is a tacit assumption that p is not too near either of the values $\pm l\beta ai$: making the contrary supposition in this case, too, I failed to obtain any evidence of the existence of a root whose real part is positive.

It may be shown that, if with the origin as centre, a circle be described for which

$$
\text{mod. } 2a\,(-p'/\nu)^{\frac{1}{2}}=(2r+1)\,\pi/2,
$$

where r is a large enough integer, the number of roots of the period-equation for which p' lies within this circle is $r-1$.* This follows as in Art. 18: the alterations in the form of the left-hand member of (72) which have to be made in different portions of the contour are, as in that Article, negligible if p' is sufficiently great.

* This is one less than if the boundary-conditions included $\nabla^2 v = 0$. (See Art. 18.)

There is obtainable as a special and limiting case the solution of the problem of the free disturbances of the fluid at rest; these have been investigated by Lord Rayleigh.* In this case, β being zero, if p' is finite u_1, u_2 are infinite, but $u_1 - u_2$ or $2ai(-p'/\nu)^{\frac{1}{2}}$ is finite; if, in (72), in the first and third expressions in $\{\quad\}$, we neglect all terms which do not involve *Exp* $(+u)$, and then equate β to zero, we obtain an equation which is valid and exact over all the plane; it may easily be verified that this equation leads to Lord Rayleigh's results.

Another special case which may be noticed is that in which λa is very great. In this case the smaller roots, i.e. those for which λ is very much greater than $\{(-p' \pm l\beta ai)/\nu\}^{\frac{1}{2}}$, are given approximately by the same formulæ as when the boundary-conditions include $S = 0$; and for those which are not so given p' is wholly real and negative. In fact, for those real values of p' which are far removed from the complex ones, the equation assumes the approximate form

$$e^{2(u_1-u_2)} \doteq \frac{[\lambda + i\{(-p' + l\beta ai)/\nu\}^{\frac{1}{2}}][\lambda + i\{(-p' - l\beta ai)/\nu\}^{\frac{1}{2}}]}{[\lambda - i\{(-p' + l\beta ai)/\nu\}^{\frac{1}{2}}][\lambda - i\{(-p' - l\beta ai)/\nu\}^{\frac{1}{2}}]}$$

$$\doteq \frac{-\lambda^2 + \nu^{-1}(p'^2 + l^2\beta^2a^2)^{\frac{1}{2}} - i\lambda\nu^{-\frac{1}{2}}\{-2p' + 2(p'^2 + l^2\beta^2a^2)^{\frac{1}{2}}\}^{\frac{1}{2}}}{-\lambda^2 + \nu^{-1}(p'^2 + l^2\beta^2a^2)^{\frac{1}{2}} + i\lambda\nu^{-\frac{1}{2}}\{-2p' + 2(p'^2 + l^2\beta^2a^2)^{\frac{1}{2}}\}^{\frac{1}{2}}}. \tag{74}$$

This equation could be solved without any great difficulty if the values of the constants were given. It will be seen that in taking successive values of p' in order of increasing magnitude, in passing through the region in which p'^2 and $\nu\lambda^2$ are of the same order, one root is, so to speak, lost as compared with the period-equation (8). All the roots of the equation (72) are thus accounted for.

In the most general case, the real values of p' which are not too near the complex ones are given by (74). As regards the determination of the complex values, though (72) simplifies somewhat, I have not been able to reduce it to a form which I can solve.

The approximate forms (72), (74), which have been obtained for the period-equation are inappropriate to small values of λa, as when λa is made equal to zero, they become identities; when λa is very small, it is more convenient to express (70) in the form

$$\int_{-a}^{a} S_1 \cosh \lambda y dy \int_{-a}^{a} S_2 \sinh \lambda y dy - \int_{-a}^{a} S_1 \sinh \lambda y dy \int_{-a}^{a} S_2 \cosh \lambda y dy = 0. \tag{75}$$

* "On the Question of the Stability of the Flow of Fluids," Phil. Mag. xxxiv., 1892, p. 59; Collected Papers, iii., p. 582.

If λa is made to diminish without limit,* this becomes

$$\int_{-a}^{a} S_1 dy \int_{-a}^{a} S_2 y dy - \int_{-a}^{a} S_1 y dy \int_{-a}^{a} S_2 dy = 0. \tag{76}$$

In the region in which the roots actually lie, this assumes the approximate form

$$\begin{aligned}\{i(6\pi)^{\frac{1}{2}} + (e^{u_1} - ie^{-u_1})\, u_1^{-\frac{1}{2}} - e^{u_2} u_2^{-\frac{1}{2}}\}\, \{e^{-u_1} u_1^{\frac{1}{6}} - e^{-u_2} u_2^{\frac{1}{6}}\} \\ - \{(e^{u_1} - ie^{-u_1})\, u_1^{\frac{1}{6}} - e^{u_2} u_2^{\frac{1}{6}}\}\, \{e^{-u_1} u_1^{-\frac{1}{2}} - e^{-u_2} u_2^{-\frac{1}{2}}\} \doteqdot 0.\end{aligned} \tag{77}$$

For real roots, if p' is not too near C (fig. 2), this may be replaced by

$$e^{u_1 - u_2} - e^{u_2 - u_1} \doteqdot 0$$

identical with (14). Even in this somewhat simple case, the equation giving the complex roots does not appear readily solvable. In this case it may be shown that the critical point at which p' becomes imaginary does not coincide with C (fig. 2); but that some of the roots become imaginary at points to the left of C, and others at points to the right; that for the roots which are of low order the absolute distance of the critical point from C is not large, and that as the order of the root rises it tends asymptotically to C. The complex roots thus consist of four series—one to the left of AC, another to the right, together with the images of these series in the axis of real quantities.

In the most general case the critical point at which roots become imaginary is not far from C; and the values of p' lie not far from the lines AC, $A'C$.

It is thus seen that, unless either λa is large, or else $\beta l a^3/\nu$ so small that all the disturbances are aperiodic, the results I have indicated are very incomplete for the natural boundary-conditions $v = 0$, $dv/dy = 0$.

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* If the velocity-gradient is great enough, λa may be very small, and yet $\beta l a^3/\nu$ not small; so that for sufficiently rapid motion this case is a little more general than that in which v is made a function of y only. In the latter case, the method similar to that of Art. 15 succeeds in proving directly that the disturbances are exponentially stable; this result was, I believe, obtained many years ago by Love.

CHAPTER III.

Applications of the Method of Osborne Reynolds.

Art. 28. *Explanation of Osborne Reynolds' Method.*

Professor Osborne Reynolds* has discussed the question of the stability of flow from a point of view very different from that adopted by Lord Kelvin. He supposes the turbulent or unstable motion to be already in existence, and seeks to determine a criterion as to whether the relative kinetic energy of the disturbed motion will increase, diminish, or remain stationary. In case the disturbance is regarded as finite, i.e. if, in the expressions for the velocities, terms of higher order than the first in small quantities are retained, the magnitudes of the velocities enter into the determining condition; but if only terms of the first order are taken into account, the criterion does not involve the scale of the disturbance, and moreover gives a lower limit than is obtainable when the disturbance is finite, for the slowest steady motion, under assigned conditions, for which a disturbance of assigned type could possibly increase. Thus the discussion of infinitesimal disturbances would appear in reality as important as that of finite ones, and is moreover considerably simpler. For infinitesimal disturbances, considering only the case in which the velocity in the steady motion is in the x-direction, and is independent of x, the criterion may be obtained as follows. Let the velocity in the steady motion be U, and that in the disturbed $U + u, v, w$, let the stress-components in the steady motion be P_{xx}, P_{xy}, etc., and those in the disturbed be $P_{xx} + p_{xx}$, $P_{xy} + p_{xy}$, etc. By writing down the fundamental equations for the disturbed and for the steady motions, and subtracting, we evidently obtain the equations

$$\begin{aligned}
du/dt + Udu/dx + vdU/dy + wdU/dz &= \rho^{-1}\{dp_{xx}/dx + dp_{xy}/dy + dp_{xz}/dz\},\\
dv/dt + Udv/dx &= \rho^{-1}\{dp_{xy}/dx + dp_{yy}/dy + dp_{yz}/dz\},\\
dw/dt + Udw/dx &= \rho^{-1}\{dp_{xz}/dx + dp_{yz}/dy + dp_{zz}/dz\}. \quad (1)
\end{aligned}$$

Multiplying by $\rho u, \rho v, \rho w$, respectively, and integrating throughout any volume, we have

$$\begin{aligned}
d/dt\,.\,\tfrac{1}{2}\int \rho(u^2 + v^2 + w^2)\, d\,.\,\mathrm{vol} = &-\int \rho u(vdU/dy + wdU/dz)\, d\,.\,\mathrm{vol}.\\
&-\frac{1}{2}\int \rho\, Ud/dx(u^2 + v^2 + w^2) + \int u\,\{dp_{xx}/dx + dp_{xy}/dy + dp_{xz}/dz\}\, d.\mathrm{vol}\\
&+ \text{two terms similar to the last.} \qquad (2)
\end{aligned}$$

* For reference, see Introduction, p. 75. An excellent résumé of Reynolds' method is contained in Lamb's "Hydrodynamics," 3rd Edition, Art. 346, from which I have paraphrased a few sentences.

On integrating by parts all the terms on the right, except the first, the right-hand member may be written

$$-\int \rho u(v\,dU/dy + w\,dU/dz)\,d.\mathrm{vol} - \frac{1}{2}\int \rho l\,U(u^2+v^2+w^2)\,dS + \int u(lp_{xx}+mp_{xy}+np_{xz})\,dS$$

$$+ \text{two terms similar to the last} - \int \{p_{xx}du/dx + p_{yy}dv/dy + p_{zz}dw/dz$$

$$+ p_{yz}(dv/dz + dw/dy) + p_{zx}(dw/dx + du/dz) + p_{xy}(du/dy + dv/dx)\}\,d.\mathrm{vol}, \quad (3)$$

dS denoting an element of the bounding surface, and l, m, n the direction-cosines of the outward drawn normal. The term involving the first surface-integral represents the rate at which kinetic energy of disturbance is convected into the volume considered, and the other three surface-terms denote the rate at which the *additional* stresses p_{xx}, p_{xy}, etc., called into existence by the disturbance, would do work in the *additional* motion u, v, w on the fluid contained in the surface. In many cases the joint effect of the surface-terms is *nil*; this happens, for instance, when the disturbance has a definite wave-length in the direction of flow, if the volume is bounded by surfaces parallel to the direction of flow, such that u, v, w vanish at them and by perpendicular planes, such that the distance between them is any multiple of a wave-length. In any such case, by substituting in the last integral in (3), the values of the stresses, viz.,

$$p_{xx} = -p - \tfrac{2}{3}\mu\,(du/dx + dv/dy + dw/dz) + 2\mu\,du/dx, \quad p_{xy} = \mu\,(du/dy + dv/dx), \text{ etc.,}$$

the right-hand member of (2) becomes

$$-\int \rho u\,(v\,dU/dy + w\,dU/dz)\,d.\mathrm{vol}$$

$$-\mu\int\{2(du/dx)^2 + 2(dv/dy)^2 + 2(dw/dz)^2 + (dv/dz+dw/dy)^2 + (dw/dx+du/dz)^2$$

$$+ (du/dy+dv/dx)^2\}\,d.\mathrm{vol} + \int p'(du/dx + dv/dy + dw/dz)\,d.\mathrm{vol}, \quad (4)$$

where $\qquad p' = p + 2\mu/3\,.\,(du/dx + dv/dy + dw/dz).$

The second member is essentially negative; the first may be either positive or negative; the third is, of course, zero, though it is convenient to retain it for the present,* thus not assuming the fluid to be incompressible; and whether the disturbance increases or decreases, depends on the sign of the whole. If then, for a given steady motion we could find the lowest value of μ for which it is possible to choose u, v, w, so that the expression (4) may be zero, there would be no possibility of the motion being unstable for a greater value of μ.

In the applications of the method by Reynolds, Sharpe, and H. A. Lorentz, the character of the disturbance is to a certain extent assumed, and apparently somewhat arbitrarily; and I proceed in the present chapter to conduct similar investigations, while endeavouring to avoid any such arbitrary choice.

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* For the purpose of variation.

ART. 29. *Differential Equations satisfied by the Disturbance which is Stationary for the Greatest Possible* μ.

Proceeding to a more general investigation, the critical equation for μ, whether the fluid be compressible or not, is from (4):

$$-\int \rho u\,(v\,dU/dy + w\,dU/dz)\,d\,.\,\mathrm{vol} + \int p'(du/dx + dv/dy + dw/dz)\,d\,.\,\mathrm{vol}$$
$$-\mu\int \{2(du/dx)^2 + 2(dv/dy)^2 + 2\,(dw/dz)^2 + (dv/dz + dw/dy)^2 + (dw/dx + du/dz)^2 + (du/dy + dv/dx)^2\}\,d\,.\,\mathrm{vol} = 0. \qquad (5)$$

The variation of u, v, w in this gives, as conditions for a stationary μ, on integrating by parts,

$$2\mu\nabla^2 u + 2\mu d/dx\,(du/dx + dv/dy + dw/dz) - \rho\,(v\,dU/dy + w\,dU/dz) = dp/dx + 4\mu/3\,.\,(du/dx + dv/dy + dw/dz), \qquad (6)$$

etc., or, supposing the fluid incompressible,*

$$\begin{aligned} 2\mu\nabla^2 u - \rho\,(v\,dU/dy + w\,dU/dz) &= dp/dx,\\ 2\mu\nabla^2 v - \rho u\,dU/dy &= dp/dy,\\ 2\mu\nabla^2 w - \rho u\,dU/dz &= dp/dz. \end{aligned} \qquad (7)$$

If the volume is bounded by fixed surfaces parallel to the direction of flow and by perpendicular planes such that the distance between them is any multiple of a wave-length, the surface terms, which have not been given, vanish; under these conditions also equations (7) with that of continuity satisfy (5), so that (5) need no longer be referred to.

ART. 30. *The uniformly Shearing Stream subject to Boundary-Conditions* $v = 0$, $dv/dy = 0$. *Lorentz' Result.*

A stream of uniform vorticity is, of course, the simplest case; and Reynolds' method has been applied to it by H. A. Lorentz.† The type of disturbance he selects consists of a species of "Elliptic Whirls" in which each particle of fluid has motion in an elliptic orbit superimposed on its steady motion; these ellipses are similar and similarly situated; and the angular velocity round the centre is a function of the distance from it; the orientation and shape of the ellipses and the law of velocity are then determined, so that the value of μ which makes the right-hand member of (4) vanish shall be greatest possible. If the steady velocity be By, and the distance between the bounding-planes D, his resulting equation is $\rho BD^2 = 288\mu$.

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* If the fluid be compressible, the variation of ρ and p in (5) leads to an equation which would determine the *scale* of the disturbance.

† "Ueber die Entstehung turbulenter Flüssigkeitsbewegungen und über den Einfluss dieser Bewegungen bei der Strömung durch Röhren." Abhandlungen über theoretische Physik, Band 1, s. 43.

Analogy with other problems leads us to assume that disturbances in two dimensions will be less stable than those in three; this view is confirmed by the corresponding result in case viscosity is neglected, seen by comparing equations (28), (38) of Part I., Chap. I.; it is further strengthened by comparing the two- and the three-dimensioned forms of equation (29), Chap. I., above, and by the discussion of the fundamental free disturbances in Chap. II. Considering, then, the two-dimensioned case,* the elimination of p from (7) gives

$$2\mu\nabla^2(du/dy - dv/dx) - \rho B(dv/dy - du/dx) = 0. \tag{8}$$

We may now conveniently introduce the stream-function ψ, when this becomes

$$\mu\nabla^2\nabla^2\psi + \rho B d^2\psi/dx\,dy = 0. \tag{9}$$

This is to be solved subject to the conditions that ψ and $d\psi/dy$ vanish at the bounding planes which we will denote by $y = \pm a$. We next suppose that, as a function of x, ψ varies as e^{ilx}, when the equation becomes

$$\mu(d^2/dy^2 - l^2)^2\psi + il\rho B d\psi/dy = 0. \tag{10}$$

The fundamental solutions are $\psi = e^{imy}$ where the values of m are given by

$$\mu(m^2 + l^2)^2 - B\rho lm = 0. \tag{11}$$

Denoting the roots of this by m_1, m_2, m_3, m_4, the equation to which the boundary conditions lead is

$$\begin{vmatrix} e^{m_1 ai} & e^{m_2 ai} & e^{m_3 ai} & e^{m_4 ai} \\ e^{-m_1 ai} & e^{-m_2 ai} & e^{-m_3 ai} & e^{-m_4 ai} \\ m_1 e^{m_1 ai} & m_2 e^{m_2 ai} & m_3 e^{m_3 ai} & m_4 e^{m_4 ai} \\ m_1 e^{-m_1 ai} & m_2 e^{-m_2 ai} & m_3 e^{-m_3 ai} & m_4 e^{-m_4 ai} \end{vmatrix} = 0, \tag{12}$$

or

$$\begin{aligned}(m_1m_2 + m_3m_4)\sin(m_1 - m_2)a\,\sin(m_3 - m_4)a \\ + (m_2m_3 + m_1m_4)\sin(m_2 - m_3)a\,\sin(m_1 - m_4)a \\ + (m_3m_1 + m_2m_4)\sin(m_3 - m_1)a\,\sin(m_2 - m_4)a = 0.\end{aligned} \tag{13}$$

As the sum of the values of m is zero, they may be written

$$p + r,\quad p - r,\quad -p + r',\quad -p - r', \tag{14}$$

where p is real, and, making these substitutions, (13) becomes

$$(4p^2 - r^2 - r'^2)\sin 2ra \sin 2r'a - 2rr'\cos 2ra\cos 2r'a + 2rr'\cos 4pa = 0. \tag{15}$$

Now, the values of m which satisfy (11) must all be imaginary, or else two real and two imaginary.

* The three-dimensioned case was attempted, but it proved too difficult.

Taking the former alternative, on writing $r = iq$, $r' = iq'$, (15) becomes

$$(4p^2 + q^2 + q'^2) \sinh 2qa \sinh 2q'a - 2qq' \cosh 2qa \cosh 2q'a + 2qq' \cos 4pa = 0. \quad (16)$$

This may be written in the form

$$(q - q')^2 \sinh^2(q + q') a - (q + q')^2 \sinh^2(q - q') a + 4p^2 \sinh 2qa \sinh 2q'a - 4qq' \sin^2 2pa = 0, \quad (17)$$

from which it is evident that it cannot be satisfied by real values of q, q'; for if they be chosen positive, as can always be done, the first term exceeds the second, and the third the fourth.

Falling back, then, on the latter alternative, and writing in (15) $r' = iq'$ simply, it becomes

$$(4p^2 + q'^2 - r^2) \sinh 2q'a \sin 2ra - 2q'r \cosh 2q'a \cos 2ra + 2q'r \cos 4pa = 0. \quad (18)$$

To find a stationary disturbance of given wave-length, and the corresponding value of μ, we have then, supposing l given, to solve the simultaneous equations involved in (18), and the statement that the values of m which satisfy (11) are $p \pm r$, $-p \pm q'i$.

Now, from the coefficients of the powers of m in (11) we have the equations

$$\begin{aligned} q'^2 - r^2 - 2p^2 &= 2l^2, \\ (p^2 + q'^2)(p^2 - r^2) &= l^4, \\ 2p(q'^2 + r^2) &= B\rho l \mu^{-1}. \end{aligned} \quad (19)$$

If we express q', r, in terms of p, l, we have

$$q'^2 = 2p\sqrt{p^2 + l^2} + p^2 + l^2, \quad (20)$$

$$r^2 = 2p\sqrt{p^2 + l^2} - p^2 - l^2, \quad (21)$$

and also obtain

$$\mu = \frac{B\rho l}{8p^2\sqrt{p^2 + l^2}}. \quad (22)$$

It may now be proved that the equation (18) has no solution for which $2ra$ is less than π. Denoting the left-hand member of that equation by V, we have

$$\begin{aligned} \tfrac{1}{2} dV/da = {} & (q'^2 + r^2)(q' \cosh 2q'a \sin 2ra - r \sinh 2q'a \cos 2ra) \\ & + 4p^2(q' \cosh 2q'a \sin 2ra + r \sinh 2q'a \cos ra) - 4pq'r \sin 4pa, \end{aligned} \quad (23)$$

$$\begin{aligned} \tfrac{1}{4} d^2V/da^2 = {} & (q'^2 + r^2)^2 \sinh 2q'a \sin 2ra \\ & + 4p^2((q'^2 - r^2) \sinh 2q'a \sin 2ra + 2q'r \cosh 2q'a \cos 2ra - 2q'r \cos 4pa), \end{aligned} \quad (24)$$

$$\tfrac{1}{8}d^3V/da^3 = (q'^2+r^2)^2\{q'\cosh 2q'a\sin 2ra + r\sinh 2q'a\cos 2ra\}$$
$$+4p^2\{(q'^3-3q'r^2)\cosh 2q'a\sin 2ra+(3q'^2r-r^3)\sinh 2q'a\cos 2ra$$
$$+4pq'r\sin 4pa\}, \qquad (25)$$

$$\tfrac{1}{16}d^4V/da^4 = (q'^2+r^2)^2\{(q'^2-r^2)\sinh 2q'a\sin 2ra+2q'r\cosh 2q'a\cos 2ra\}$$
$$+4p^2\{(q'^4-6q'^2r^2+r^4)\sinh 2q'a\sin 2ra+4q'r(q'^2-r^2)\cosh 2q'a\cos 2ra$$
$$+8p^2q'r\cos 4pa\}. \qquad (26)$$

When a is zero, the first three differential coefficients vanish, and the fourth is positive. Substituting the values of q', r, given by (20) and (21), (26) gives

$$\tfrac{1}{16}d^4V/da^4 = 64p^2l^2(p^2+l^2)\sinh 2q'a\sin 2ra$$
$$+64p^2(p^2+l^2)^{\frac{3}{2}}(3p^2-l^2)^{\frac{1}{2}}\cosh 2q'a\cos 2ra$$
$$+32p^4(p^2+l^2)^{\frac{1}{2}}(3p^2-l^2)^{\frac{1}{2}}\cos 4pa. \qquad (27)$$

This cannot vanish for any value of $2ra$ less than $\pi/3$; since for such values the second term exceeds the third even on replacing $\cos 4pa$ by -1, and since the first term is positive. Therefore, neither can V itself vanish, if $2ra < \pi/3$. Again, V may be written

$$(6p^2+2l^2)\sinh 2q'a\sin 2ra - 2(p^2+l^2)^{\frac{1}{2}}(3p^2-l^2)^{\frac{1}{2}}\cosh 2q'a\cos 2ra$$
$$+2(p^2+l^2)^{\frac{1}{2}}(3p^2-l^2)^{\frac{1}{2}}\cos 4pa, \qquad (28)$$

which, when $\sin 2ra$ is positive, is algebraically greater than

$$2(p^2+l^2)^{\frac{1}{2}}(3p^2-l^2)^{\frac{1}{2}}\{3^{\frac{1}{2}}\sinh 2q'a\sin 2ra - \cosh 2q'a\cos 2ra + \cos 4pa\}. \qquad (29)$$

Of the terms in brackets, when $2ra$ lies between $\pi/3$ and $\pi/2$, the first term is greater than $\tfrac{3}{2}\sinh 2q'a$; the second is numerically less than $\tfrac{1}{2}\cosh 2q'a$; and thus the three are algebraically greater than $\tfrac{3}{2}\sinh 2q'a - \tfrac{1}{2}\cosh 2q'a - 1$, and, as $q' > r\sqrt{3}$, this is certainly positive. And, since $q' > r\sqrt{3}$, it is evident that (29) cannot vanish if $2ra$ lies between $\pi/2$ and π. Thus (18) has no solution for which $2ra < \pi$.

When $2ra > \pi$, $\sinh 2q'a$ and $\cosh 2q'a$ each exceed 100; and accordingly in (18) we may neglect the term involving $\cos 4pa$, and may equate $\sinh 2q'a$ and $\cosh 2q'a$; the equation thus sensibly becomes, making use of (28),

$$\tan 2ra \doteq (p^2+l^2)^{\frac{1}{2}}(3p^2-l^2)^{\frac{1}{2}}(3p^2+l^2)^{-1}. \qquad (30)$$

The simultaneous equations (21), (30) have, of course, an infinity of solutions; there is one for which $2ra$ lies between π and $4\pi/3$; it may be shown that there is only one; for, by the aid of (21), we may write (30) in the form

$$r^{-1}\tan 2ra = (2p\sqrt{p^2+l^2}+p^2+l^2)^{\frac{1}{2}}(3p^2+l^2)^{-1}; \qquad (31)$$

as p increases beyond the value $l/\sqrt{3}$, the right-hand member continually decreases, while the left-hand member continually increases, for r, given by (21), continually increases. And it is this solution which we require; for (21), (22) show that, l being given, the smallest value of r corresponds to the largest value of μ for which the disturbance could possibly increase.

We finally wish to obtain the greatest value which the value of μ so found can be made to assume by varying l. A stationary μ is a maximum μ, for μ has no minimum; as l increases indefinitely, r remains finite, ra being $< 4\pi/3$, and p, satisfying (21), tends to equality with $l/\sqrt{3}$, so that μ given by (22) diminishes indefinitely. The differentiation of (22) gives us for a stationary μ

$$p^3 dl/dp = (3p^2 + 2l^2)\,l. \tag{32}$$

By differentiating (30), making use of this, we obtain

$$ap^3(3p^2 + 2l^2)(3p^2 - l^2)^{\frac{1}{2}}\,dr/dp \doteq -2l^4(p^2 + l^2)^{\frac{1}{2}}; \tag{33}$$

and in a similar manner from (21),

$$p^3 r\,dr/dp = 2p(p^2 + l^2)^{\frac{3}{2}} - (p^2 + l^2)(p^2 + 2l^2). \tag{34}$$

Combining (33) and (34), there results

$$a(3p^2 + 2l^2)(3p^2 - l^2)^{\frac{1}{2}}\{(p^2 + 2l^2)(p^2 + l^2)^{\frac{1}{2}} - 2p(p^2 + l^2)\} \doteq 2l^4 r, \tag{35}$$

and this, (21), and (30) are equations determining l, p, r. From (21) and (35) we obtain

$$2ra\,(3p^2 + 2l^2)\{p^2 + 2l^2 - 2p\,(p^2 + l^2)^{\frac{1}{2}}\} \doteq 4l^4\{2p - (p^2 + l^2)^{\frac{1}{2}}\}(3p^2 - l^2)^{-\frac{1}{2}}. \tag{36}$$

If $2ra$ were $7\pi/6$, the value of l^2/p^2 which would satisfy this would be ·93; while, if $2ra$ were π, it would be ·94. It will be seen that the former supposition is very nearly correct; taking then the former value of l^2/p^2, substitution in (30) shows that $2ra$ is the circular measure of 206° 57′ (the latter would give about 3′ less), i.e. $2ra = 3{\cdot}61$. From (21) there is next obtained $l/r = 1{\cdot}05$ (and $< 1{\cdot}06$), giving $la = 1{\cdot}89$. Then (22) gives

$$B\rho/(8r^2\mu) = p^2 l^{-1}\{2p - \sqrt{p^2 + l^2}\}^{-1} = 1{\cdot}698 \text{ (and } < 1{\cdot}699). \tag{37}$$

Thus, if $D = 2a$, the distance between the bounding planes, there finally results

$$B\rho a^2/\mu \doteq 44{\cdot}3 \quad \text{or} \quad B\rho D^2/\mu \doteq 177. \tag{38}$$

This result has been obtained on the supposition that the initial disturbance has a definite, but undetermined, wave-length; but as the different wave-lengths contribute to the rate of increase of the energy of disturbance terms which are simply additive, this restriction may be removed, provided the proper end-conditions are satisfied, and for this it is sufficient that on every stream-line the end-values of the velocities and of the alteration in pressure should be the same.

ART. 31. *Two instances of other Boundary-Conditions.*

As another example, suppose the former boundary-conditions are replaced by $v = 0$, $d^2v/dy^2 = 0$, equivalent to $\psi = 0$, $d^2\psi/dy^2 = 0$. Equation (13) has now to be replaced by

$$(m_1^2m_2^2 + m_3^2m_4^2)\sin(m_1 - m_2)a\sin(m_3 - m_4)a + (m_2^2m_3^2 + m_1^2m_4^2)\sin(m_2 - m_3)a \sin(m_1 - m_4)a + (m_3^2m_1^2 + m_2^2m_4^2)\sin(m_3 - m_1)a\sin(m_2 - m_4)a = 0, \quad (39)$$

or, in the notation of (14),

$$\{(r^2 - r'^2)^2 - 4p^2(r^2 + r'^2)\}\sin 2ra\sin 2r'a + 8p^2rr'\cos 2ra\cos 2r'a - 8p^2rr'\cos 4pa = 0. \quad (40)$$

On writing again $r = iq$, $r' = iq'$, this becomes

$$\{(q^2 - q'^2)^2 + 4p^2(q^2 + q'^2)\}\sinh 2qa\sinh 2q'a + 8p^2qq'\cosh 2qa\cosh 2q'a - 8p^2qq'\cos 4pa = 0. \quad (41)$$

As the first two terms are positive, and the second exceeds the third numerically, this equation cannot be satisfied, and, accordingly, as before, we fall back on the other alternative, viz., r real and r' imaginary. Writing in (40) $r' = iq'$ simply, it becomes

$$\{(r^2 + q'^2)^2 + 4p^2(q'^2 - r^2)\}\sinh 2q'a\sin 2ra + 8p^2q'r\cosh 2q'a\cos 2ra - 8p^2q'r\cos 4pa = 0. \quad (42)$$

Now this equation has no solution for which $2ra$ is less than $\pi/2$; for within this limit, as $q'^2 > 3r^2$, the left-hand member is certainly algebraically greater than

$$8p^2r\{r\sinh 2q'a\sin 2ra + q'\cosh 2q'a\cos 2ra - q'\cos 4pa\}; \quad (43)$$

and while $2ra$ increases from 0 to $\pi/2$, the sum of the first and second terms in the brackets increases continually, and therefore everywhere exceeds its initial value q'; hence the result follows. We may, therefore, equate $\sinh 2q'a$ and $\cosh 2q'a$, and neglect $\cos 4pa$ in comparison. Thus we have, expressing the coefficients in terms of p, l,

$$\tan 2ra \doteqdot -\tfrac{1}{3}(3p^2 - l^2)^{\frac{1}{2}}(p^2 + l^2)^{-\frac{1}{2}}, \quad (44)$$

and the lowest value of $2ra$ accordingly lies between $5\pi/6$ and π. As a condition for a stationary value of μ, we now obtain, using (32),

$$ap(3p^2 + 2l^2)(3p^2 - l^2)^{\frac{1}{2}}\,dr/dp = 3l^2(p^2 + l^2)^{\frac{1}{2}}, \quad (45)$$

and, by the aid of (21), (32), (34), there results, instead of (36), the equation

$$2ar(3p^2 + 2l^2)(2p(p^2 + l^2)^{\frac{1}{2}} - p^2 - 2l^2) = 6p^2l^2(2p - (p^2 + l^2)^{\frac{1}{2}})(3p^2 - l^2)^{-\frac{1}{2}}. \quad (46)$$

Substituting $2ar = 5\pi/6, \pi$ we obtain $l^2/p^2 = \cdot 73, \cdot 75$, respectively. The former value substituted in (44) gives $2ra$ to be less than π by the circular measure of 20° 54′; and the latter 20° 42′; we therefore see that the correct value of l^2/p^2 is nearly ·736, and that of the angle in question 20° 50′; thus $2ar = 2{\cdot}778$, and finally

$$B\rho a^2/\mu = 26{\cdot}36 \quad \text{or} \quad B\rho D^2/\mu \doteqdot 105{\cdot}5. \tag{47}$$

If, again, we were to take as boundary-conditions

$$dv/dy = 0, \quad d^2v/dy^2 = 0,$$

we should obtain equation (13) over again, and the same criterion as in (38).

Art. 32. *A Stream between fixed Parallel Planes. Results of Reynolds and of Sharpe.*

The case of flow between fixed parallel planes was the only one to which Reynolds himself applied his method so as to obtain a numerical result.* Noting that if the disturbance is expressed as a trigonometrical function of y, the higher harmonics would, on the whole, make for increased stability, he chose as the type to be investigated one in which

$$u = A(\cos p + 3\cos 3p)\cos \pi lx/2a + B(2\cos 2p + 2\cos 4p)\sin \pi lx/2a, \tag{48}$$
$$v = lA(\sin p + \sin 3p)\sin \pi lx/2a - lB(\sin 2p + 2^{-1}\sin 4p)\cos \pi lx/2a, \tag{49}$$

where $p = \pi y/2a$. The values of l and of B/A were then so determined that the value of μ obtained by equating to zero the rate of increase of the energy of disturbance should be greatest possible, and the result he obtained for the critical equation was

$$D\overline{U}\rho/\mu = 517, \tag{50}$$

where $D = 2a$, the distance between the planes, and $\overline{U}$ is the mean velocity.

This case has also been discussed by Sharpe;† he chose as the type of disturbance that in which, in the same notation,

$$u = A(\sin p + \sin 3p)\cos \pi lx/2a + B(2\sin 2p + 4\sin 4p)\sin \pi lx/2a, \tag{51}$$
$$v = -lA(\cos p + 3^{-1}\cos 3p)\sin \pi lx/2a + lB(\cos 2p + \cos 4p)\cos \pi lx/2a, \tag{52}$$

and obtained a lower value for the critical velocity, his equation being

$$D\overline{U}\rho/\mu = 167. \tag{53}$$

* Loc. cit., p. 75, ante.

† "On the Stability of the Motion of a Viscous Liquid": Trans. Amer. Math. Soc., vol. vi. No. 4, October, 1905.

ART. 33. *The more General Investigation.*

Proceeding to a more general investigation, if the axis of x be taken midway between the planes, and the steady velocity be $U = C(a^2 - y^2)$, and keeping to the two-dimensioned case, equations (7) are replaced by

$$2\mu\nabla^2 u + 2C\rho yv = dp/dx,$$

$$2\mu\nabla^2 v + 2C\rho yu = dp/dy. \tag{54}$$

Eliminating p, and substituting for U, we obtain

$$2\mu\nabla^2(du/dy - dv/dx) + 2C\rho\{y(dv/dy - du/dx) + v\} = 0, \tag{55}$$

or, introducing the stream function, ψ,

$$\mu\nabla^4\psi - C\rho\{2yd^2\psi/dxdy + d\psi/dx\} = 0. \tag{56}$$

If we now further suppose that ψ varies as e^{ilx}, where l is definite, but undetermined, this is reduced to

$$\mu(d^2/dy^2 - l^2)^2\psi - C\rho li(2yd\psi/dy + \psi) = 0. \tag{57}$$

It seems convenient to substitute $ly = \alpha$, $C\rho i/\mu l^3 = k$, and doing so this equation becomes

$$(d^2/d\alpha^2 - 1)^2\psi - k(2\alpha d\psi/d\alpha + \psi) = 0. \tag{58}$$

This can be solved in series preceding in ascending powers of α. Writing

$$\psi = \Sigma A_n\alpha^n/\underline{|n}, \tag{59}$$

the coefficient law is

$$A_{n+4} - 2A_{n+2} + \{1 - (2n+1)k\}A_n = 0. \tag{60}$$

There are, therefore, series whose first terms are respectively 1, α, α^2, α^3. If u, v, or ψ, $d\psi/dy$ are to vanish at the boundaries $y = \pm a$, there is evidently one solution of the problem in which ψ is an even function of y, and another in which it is odd. And there are various reasons for supposing that the former, i.e., that in which v is an even, and u an odd, function, will give the narrower limit of stability. This view is in conformity with the fact that Sharpe obtained a lower value for $D\overline{U}\rho/\mu$ than Reynolds did; I understand Sharpe to state that it seems more in accordance with experiments that v should have a maximum midway between the planes than that u should; and I obtained this result when la is very small.

When la is sufficiently small, we may replace the coefficient law (60) by the simpler one

$$A_{n+4} - (2n+1)kA_n = 0. \tag{61}$$

The values of ψ then proceed simply in powers of $k\boldsymbol{a}^4$, all other terms being omitted. Equation (68) given by the boundary-conditions becomes

$$1 + \frac{32k^2\boldsymbol{a}_1^{8}}{\lfloor\underline{9}} + \frac{15360k^4\boldsymbol{a}_1^{16}}{\lfloor\underline{17}} + \frac{1426^{*}\, k^6\boldsymbol{a}_1^{24}}{10^{21}} + \ldots = 0.\dagger \qquad (62)$$

The lowest root of this is approximately

$$C\rho l a^4/\mu = -\, ik\boldsymbol{a}_1^{4} \doteqdot 107. \qquad (63)$$

On the other hand, the odd forms of ψ lead to the equation

$$1 + \frac{k^2\boldsymbol{a}_1^{8}}{69300} + \frac{31815^{*}}{10^{16}}\, k^4\boldsymbol{a}_1^{16} + \frac{625^{*}}{10^{22}}\, k^6\boldsymbol{a}_1^{24} + \ldots = 0, \qquad (64)$$

and the lowest root gives approximately

$$C\rho l a^4/\mu = -\, ik\boldsymbol{a}_1^{4} \doteqdot 265. \qquad (65)$$

Considering then the even forms of ψ, one of the series whose lowest term is unity is

$$\begin{aligned}\psi_0\ddagger = 1 &+ \frac{2\boldsymbol{a}^2}{\lfloor\underline{2}} + (3+k)\frac{\boldsymbol{a}^4}{\lfloor\underline{4}} + (4+12k)\frac{\boldsymbol{a}^6}{\lfloor\underline{6}} + (5+50k+9k^2)\frac{\boldsymbol{a}^8}{\lfloor\underline{8}} + (6+140k+174k^2)\frac{\boldsymbol{a}^{10}}{\lfloor\underline{10}} \\ &+ (7+315k+1189k^2+9.17k^3)\frac{\boldsymbol{a}^{12}}{\lfloor\underline{12}} + (8+616k+5144k^2+3960k^3)\frac{\boldsymbol{a}^{14}}{\lfloor\underline{14}} \\ &+ (9+1092k+16974k^2+37492k^3+9.17.25k^4)\frac{\boldsymbol{a}^{16}}{\lfloor\underline{16}} + (\ldots 122490k^4)\frac{\boldsymbol{a}^{18}}{\lfloor\underline{18}} + \ldots\end{aligned} \qquad (66)$$

and that whose lowest term is $\boldsymbol{a}^2/2$ is

$$\begin{aligned}\psi_2 = &\frac{\boldsymbol{a}^2}{\lfloor\underline{2}} + \frac{2\boldsymbol{a}^4}{\lfloor\underline{4}} + (3+5k)\frac{\boldsymbol{a}^6}{\lfloor\underline{6}} + (4+28k)\frac{\boldsymbol{a}^8}{\lfloor\underline{8}} + (5+90k+5.13k^2)\frac{\boldsymbol{a}^{10}}{\lfloor\underline{10}} \\ &+ (6+220k+606k^2)\frac{\boldsymbol{a}^{12}}{\lfloor\underline{12}} + (7+455k+3037k^2+5.13.21k^3)\frac{\boldsymbol{a}^{14}}{\lfloor\underline{14}} \\ &+ (8+840k+10968k^2+17880k^3)\frac{\boldsymbol{a}^{16}}{\lfloor\underline{16}} \\ &\qquad + (9+1428k+32094k^2-122468k^3+5.13.21.29k^4)\frac{\boldsymbol{a}^{18}}{\lfloor\underline{18}} + \ldots \\ &\qquad + (\ldots 669210k^4)\frac{\boldsymbol{a}^{20}}{\lfloor\underline{20}} + \ldots\end{aligned} \qquad (67)$$

The boundary-conditions $u = 0,\ v = 0$ evidently give

$$\psi_0\, d\psi_2/d\boldsymbol{a} - \psi_2\, d\psi_0/d\boldsymbol{a} = 0, \qquad (68)$$

* These numbers are only approximately correct.

† The boundary-value of $\boldsymbol{a}$ is denoted by $\boldsymbol{a}_1$.

‡ Probably the numerical work would have been simpler had I chosen $\psi_0 - \psi_2$, instead of ψ_0.

where, in determining a, y is equated to α. Denoting this boundary-value of a by a_1, this equation, after division by a_1, becomes

$$1 + \frac{2a_1^2}{\lfloor\underline{3}} + \frac{8a_1^4}{\lfloor\underline{5}} + \frac{32a_1^6}{\lfloor\underline{7}} + (128 + 32k^2)\frac{a_1^8}{\lfloor\underline{9}} + (512 + 320k^2)\frac{a_1^{10}}{\lfloor\underline{11}}$$
$$+ (2048 + 2816k^2)\frac{a_1^{12}}{\lfloor\underline{13}} + (8192 + 128.168k^2)\frac{a_1^{14}}{\lfloor\underline{15}}$$
$$+ (32768 + 147456k^2 + 15360k^4)\frac{a_1^{16}}{\lfloor\underline{17}} + \ldots$$
$$+ (131072 + 32^2.912k^2 + 276480k^4)\frac{a_1^{18}}{\lfloor\underline{19}} + \ldots = 0 \qquad (69)$$

In verification of the somewhat lengthy numerical work involved in calculating the coefficients in (69), I obtained it as far as the terms involving k^2 in another way, using solutions of (58) in the form of series which proceed in ascending powers of k, the coefficient of each power being a function of a. This method did not appear to have much advantage over the other. The portion of the left-hand member of (69) which is independent of k is

$$(2a_1 + \sinh 2a_1)/4a_1.$$

We have now, regarding l, and therefore a_1, as given, to solve (69), choosing the highest root in μ, and therefore the lowest value of k. Then l has to be chosen, so that this value of μ is the greatest possible, i.e. the lowest value of $-ika_1^3$ is to be made a minimum. The lowest value of $-k^2a_1^6$ is, approximately,

$$-k^2a_1^6 \doteqdot 1 + \frac{2a_1^2}{\lfloor\underline{3}} + \frac{8a_1^4}{\lfloor\underline{5}} + \frac{32a_1^6}{\lfloor\underline{7}} + \frac{128a_1^8}{\lfloor\underline{9}} + \frac{512a_1^{10}}{\lfloor\underline{11}} + \frac{2048a_1^{12}}{\lfloor\underline{13}} + \ldots$$
$$\div 32\left\{\frac{a_1^2}{\lfloor\underline{9}} + \frac{10a_1^4}{\lfloor\underline{11}} + \frac{88a_1^6}{\lfloor\underline{13}} + \frac{672a_1^8}{\lfloor\underline{15}} + \frac{4608a_1^{10}}{\lfloor\underline{17}} + \frac{21504a_1^{12}}{\lfloor\underline{19}} + \ldots\right\} \qquad (70)$$

in which terms involving k^4 have been neglected. Making this stationary, we obtain the equation

$$1 + \frac{2a_1^2}{\lfloor\underline{3}} + \frac{8a_1^4}{\lfloor\underline{5}} + \frac{32a_1^6}{\lfloor\underline{7}} + \frac{128a_1^8}{\lfloor\underline{9}} + \frac{512a_1^{10}}{\lfloor\underline{11}} + \frac{2048a_1^{12}}{\lfloor\underline{13}} + \ldots$$
$$\div \left\{\frac{a_1^2}{\lfloor\underline{9}} + \frac{10a_1^4}{\lfloor\underline{11}} + \frac{88a_1^6}{\lfloor\underline{13}} + \frac{672a_1^8}{\lfloor\underline{15}} + \frac{4608a_1^{10}}{\lfloor\underline{17}} + \frac{29184a_1^{12}}{\lfloor\underline{19}} + \ldots\right\}$$
$$= \frac{2}{\lfloor\underline{3}} + \frac{2.8a_1^2}{\lfloor\underline{5}} + \frac{3.32a_1^4}{\lfloor\underline{7}} + \frac{4.128a_1^6}{\lfloor\underline{9}} + \frac{5.512a_1^8}{\lfloor\underline{11}} + \frac{6.2048a_1^{10}}{\lfloor\underline{13}} + \ldots$$
$$\div \left\{\frac{1}{\lfloor\underline{9}} + \frac{2.10a_1^2}{\lfloor\underline{11}} + \frac{3.88a_1^4}{\lfloor\underline{13}} + \frac{4.672a_1^6}{\lfloor\underline{15}} + \frac{5.4608a_1^8}{\lfloor\underline{17}} + \frac{6.29184a_1^{10}}{\lfloor\underline{19}} + \ldots\right\} \qquad (71)$$

which reduces to

$$1 + \frac{2}{11}\alpha_1^2 - \frac{3}{11.13}\alpha_1^4 - \frac{32.961.\lfloor 9}{\lfloor 15}\alpha_1^6 - \frac{512.6223.\lfloor 9}{3\lfloor 17}\alpha_1^8 - \frac{512.802.\lfloor 9}{5\lfloor 17}\alpha_1^{10} \ldots = 0. \tag{72}$$

This has a root in the neighbourhood of $\alpha_1^2 = 4{\cdot}4$. The minimum value of $-k^2\alpha_1^6$ is by no means sharply defined; the values 4·3, 4·4, 4·5 substituted in (70) give $-k^2\alpha_1^6 = 7591,\ 7565,\ 7576$ respectively. These all give

$$C\rho a^3/\mu = -ika^3 \doteq 87. \tag{73}$$

In (70), however, the terms involving the fourth and higher powers of k have been neglected. If we substitute the values which have been found for k and α_1 in the two terms involving k^4 in (69) the former would raise the value of $-k^2\alpha_1^6$ by about 1 per cent., and the latter by about one-fourth as much. We would presumably make proper allowance for all the terms neglected* if we increase the value found for $-k^2\alpha_1^6$ by 2 per cent., or that of $-ika_1^3$ by 1 per cent.. Thus we would obtain the criterion

$$D\overline{U}\rho/\mu = 4C\rho a^3/3\mu = 117. \tag{74}$$

ART. 34. *Flow through a Circular Pipe. Sharpe's Result.*

The case also of flow through a circular pipe has been discussed by Sharpe.† Taking the z axis in the direction of flow, he selected an initial disturbance in which

$$\begin{aligned} 2au &= lA\pi r(\sin p + \sin 3p)\sin \pi lz/2a - lB\pi r(\sin p + 2^{-1}\sin 4p)\cos \pi lz/2a, \\ 2aw &= A\{4a(\sin p + \sin 3p) + \pi r(\cos p + 3\cos 3p)\}\cos \pi lz/2a \\ &\quad + B\{4a(\sin 2p + 2^{-1}\sin 4p) + \pi r(2\cos 2p + 2\cos 4p)\}\sin \pi lz/2a, \end{aligned} \tag{75}$$

where u is measured radially, w in the direction of flow, the radius is a, and p denotes $\pi r/2a$. On investigating the values of B/A and of l, which lead to the greatest possible value of μ for which the disturbance could be stationary, he arrived at the equation

$$D\rho\overline{W}/\mu = 2a\rho\overline{W}/\mu = 470, \tag{76}$$

$\overline{W}$ being the mean velocity in the steady motion. I believe, however, that his work contains a numerical error‡ which sensibly affects the result; and that if this were corrected, the number 470 would be reduced to about 363.

* It appears that we may safely neglect terms in which occur k^6 or higher powers; for the left-hand member of (62) forms part of the left-hand member of (69); as far as can be judged, the term involving k^6 in the former is the most important term involving it in the latter; and substitution of the numbers just found shows the value of this term to be about 1/20000.

† Loc. cit.

‡ A coefficient of B^2l^2 in a certain equation which Sharpe gives as 6·67 should, I think, be

$$(\pi^4 - 275\pi^2/24 + 1312/27)/16 \text{ or } 2{\cdot}057.$$

ART. 35. *A circular Pipe; the more General Investigation.*

In discussing the most general disturbance in this case, we may either transform to cylindrical coordinates the equation (5), and obtain in those coordinates the equations giving a stationary μ, or else obtain in Cartesian coordinates the equations which would now replace (7), and then transform them. Adopting the latter procedure, the equations are

$$2\mu\nabla^2 u_x - \rho w dW/dx = dp/dx$$

$$2\mu\nabla^2 u_y - \rho w dW/dy = dp/dy,$$

$$2\mu\nabla^2 w - \rho(u_x dW/dx + u_y dW/dy) = dp/dz, \qquad (77)$$

where u_x, u_y denote the velocity-components in the x, y directions transverse to that of flow. Confining ourselves to the symmetrical case, which there is little doubt will give the lowest critical velocity, we write

$$u_x = xu/r, \quad u_y = yu/r,$$

when the two former equations become

$$2\mu(\nabla^2 u - ur^{-2}) - \rho w dW/dr = dp'/dr, \qquad (78)$$

and the latter is

$$2\mu\nabla^2 w - \rho u dW/dr = dp'/dz. \qquad (79)$$

Noting that

$$d/dr \,.\, \nabla^2 = (\nabla^2 - r^{-2})\, d/dr, \qquad (80)$$

and writing $W = C'(a^2 - r^2)$, the elimination of p between these gives

$$2\mu(\nabla^2 - r^{-2})(du/dz - dw/dr) + 2C'\rho\{r(dw/dz - du/dr) - u\} = 0, \quad (81)$$

Introducing the stream-function ψ defined by the equations

$$ru = d\psi/dz, \quad rw = -d\psi/dr,$$

this becomes

$$\mu(\nabla^2 - r^{-2})\{r^{-1}(d^2\psi/dr^2 + d^2\psi/dz^2) - r^{-2}d\psi/dr\} - 2C'\rho d^2\psi/drdz = 0;$$

or, $$\mu r^{-1}\{d^2/dr^2 - r^{-1}d/dr + d^2/dz^2\}^2\psi - 2C'\rho d^2\psi/drdz = 0. \qquad (82)$$

[On multiplying by r, differentiating with respect to r, and dividing by r, this might be written

$$\mu\nabla^4 w - 2C'\rho r^{-1} d^2(r^2 w)/drdz = 0, \qquad (83)$$

an equation which might be obtained more easily directly from the equations which replace (7). In the subsequent investigation, w might equally well be taken as the unknown function, instead of ψ.]

We next suppose that, as a function of z, ψ varies as e^{ilz}; then (82) is equivalent to

$$\mu\{d^2/dr^2 - r^{-1}d/dr - l^2\}^2\psi - 2C'l\rho i r d\psi/dr = 0. \tag{84}$$

It will now be convenient to substitute

$$lr = 2a, \quad 2C'\rho i/\mu l^3 = k, \tag{85}$$

when the equation becomes

$$(d^2/da^2 - a^{-1}d/da - 4)^2\psi - 16ka d\psi/da = 0. \tag{86}$$

Solving this in a series of the form

$$\psi = \Sigma A_n a^n = \Sigma \frac{B_n a^n}{\lfloor\frac{n}{2}\,\lfloor\frac{n}{2}-1},$$

the law connecting coefficients is

$$(n+4)(n+2)^2 nA_{n+4} - 8(n+2)nA_{n+2} + 16(1-nk)A_n = 0, \tag{87}$$

or

$$B_{n+4} - 2B_{n+2} + (1-nk)B_n = 0. \tag{88}$$

There are evidently solutions whose initial terms are respectively 1, a^2, $a^2 \log a$, a^4. As ψ/r and $r^{-1}d\psi/dr$ must be finite when r vanishes, the solutions with which we are concerned are those whose first terms are a^2, a^4.

The latter is

$$\begin{aligned}\psi_4 = {} & \frac{a^4}{\lfloor 2} + 2\frac{a^6}{\lfloor 2\lfloor 3} + (3+4k)\frac{a^8}{\lfloor 3\lfloor 4} + (4+20k)\frac{a^{10}}{\lfloor 4\lfloor 5} + (5+60k+32k^2)\frac{a^{12}}{\lfloor 5\lfloor 6} \\ & + (6+140k+264k^2)\frac{a^{14}}{\lfloor 6\lfloor 7} + (7+280k+1216k^2+384k^3)\frac{a^{16}}{\lfloor 7\lfloor 8} \\ & + (8+504k+4128k^2+4464k^3)\frac{a^{18}}{\lfloor 8\lfloor 9} \\ & + (9+840k+11520k^2+28000k^3+6144k^4)\frac{a^{20}}{\lfloor 9\lfloor 10} \\ & + (10+1320k+27984k^2+125840k^3+92640k^4)\frac{a^{22}}{\lfloor 10\lfloor 11} \\ & + (\ldots+739136k^4+122880k^5)\frac{a^{24}}{\lfloor 11\lfloor 12} + (\ldots+2283840k^5)\frac{a^{26}}{\lfloor 12\lfloor 13} \\ & + (\ldots+4.8.12.16.20.24k^6)\frac{a^{28}}{\lfloor 13\lfloor 14} + \ldots \end{aligned} \tag{89}$$

One of the former is

$$\begin{aligned}\psi_2 = {} & a^2 + \frac{a^4}{\lfloor\underline{2}} + (1+2k)\frac{a^6}{\lfloor\underline{2}\lfloor\underline{3}} + (1+8k)\frac{a^8}{\lfloor\underline{3}\lfloor\underline{4}} + (1+20k+12k^2)\frac{a^{10}}{\lfloor\underline{4}\lfloor\underline{5}} \\ & + (1+40k+88k^2)\frac{a^{12}}{\lfloor\underline{5}\lfloor\underline{6}} + (1+70k+364k^2+120k^3)\frac{a^{14}}{\lfloor\underline{6}\lfloor\underline{7}} \\ & + (1+112k+1120k^2+1296k^3)\frac{a^{16}}{\lfloor\underline{7}\lfloor\underline{8}} + (1+168k+2856k^2+7568k^3+1680k^4)\frac{a^{18}}{\lfloor\underline{8}\lfloor\underline{9}} \\ & + (1+240k+6384k^2+31760k^3+24096k^4)\frac{a^{20}}{\lfloor\underline{9}\lfloor\underline{10}} \\ & + (\ldots+182736k^4+30240k^5)\frac{a^{22}}{\lfloor\underline{10}\lfloor\underline{11}} + (\ldots+542400k^5)\frac{a^{24}}{\lfloor\underline{11}\lfloor\underline{12}} \\ & + (\ldots+2.6.10.14.18.22k^6)\frac{a^{26}}{\lfloor\underline{12}\lfloor\underline{13}} + \ldots \end{aligned} \tag{90}$$

The boundary-conditions $u = 0,\ v = 0$ evidently give

$$\psi_2 d\psi_4/da - \psi_4 d\psi_2/da = 0, \tag{91}$$

where, in a, r is equated to a. Denoting this value of a by a_1, this equation, on division by $a_1{}^4$, becomes

$$\begin{aligned} & \frac{1}{2} + \frac{2a_1{}^2}{\lfloor\underline{3}} + \frac{5a_1{}^4}{\lfloor\underline{2}\lfloor\underline{4}} + \frac{14a_1{}^6}{\lfloor\underline{3}\lfloor\underline{5}} + (42+4k^2)\frac{a_1{}^8}{\lfloor\underline{4}\lfloor\underline{6}} + (132+40k^2)\frac{a_1{}^{10}}{\lfloor\underline{5}\lfloor\underline{7}} \\ & + (429+280k^2)\frac{a_1{}^{12}}{\lfloor\underline{6}\lfloor\underline{8}} + (1430+1680k^2)\frac{a_1{}^{14}}{\lfloor\underline{7}\lfloor\underline{9}} + (4862+9240k^2+336k^4)\frac{a_1{}^{16}}{\lfloor\underline{8}\lfloor\underline{10}} \\ & + (16796+48048k^2+6048k^4)\frac{a_1{}^{18}}{\lfloor\underline{9}\lfloor\underline{11}} + (\ldots+55684k^4)\frac{a_1{}^{20}}{\lfloor\underline{10}\lfloor\underline{12}} \\ & + (\ldots k^4)\frac{a_1{}^{22}}{\lfloor\underline{11}\lfloor\underline{13}} + (\ldots+95040k^6)\frac{a_1{}^{24}}{\lfloor\underline{12}\lfloor\underline{14}} + \ldots = 0. \end{aligned} \tag{92}$$

[The terms on the left which are independent of k are those of

$$2a_1{}^{-4}\int_0^{a_1} a\,\{I_1(2a)\}^2 da.]$$

The lowest value of $-k^2a_1{}^6$ is therefore approximately

$$\begin{aligned} -k^2a_1{}^6 \doteqdot {} & \frac{1}{2} + \frac{2a_1{}^2}{\lfloor\underline{3}} + \frac{5a_1{}^4}{\lfloor\underline{2}\lfloor\underline{4}} + \frac{14a_1{}^6}{\lfloor\underline{3}\lfloor\underline{5}} + \frac{42a_1{}^8}{\lfloor\underline{4}\lfloor\underline{6}} + \frac{132a_1{}^{10}}{\lfloor\underline{5}\lfloor\underline{7}} + \frac{429a_1{}^{12}}{\lfloor\underline{6}\lfloor\underline{8}} + \frac{1430a_1{}^{14}}{\lfloor\underline{7}\lfloor\underline{9}} + \ldots \\ & \div \left\{\frac{4a_1{}^2}{\lfloor\underline{4}\lfloor\underline{6}} + \frac{40a_1{}^4}{\lfloor\underline{5}\lfloor\underline{7}} + \frac{280a_1{}^6}{\lfloor\underline{6}\lfloor\underline{8}} + \frac{1680a_1{}^8}{\lfloor\underline{7}\lfloor\underline{9}} + \frac{9240a_1{}^{10}}{\lfloor\underline{8}\lfloor\underline{10}} + \frac{48048a_1{}^{12}}{\lfloor\underline{9}\lfloor\underline{11}} + \ldots\right\}, \end{aligned} \tag{93}$$

in which terms involving k^4 have been neglected. We have then to choose a_1 so that this value shall be least possible. The requisite value of a_1 is not well defined, but is in the neighbourhood of 3·7. Substitutions of $a_1^2 = 3{\cdot}5, 3{\cdot}7, 4$ in (93) give respectively $-k^2a_1^6 = 1940, 1938, 1946$. In these, however, the terms involving k^4 in (92) have been neglected. If we substitute the approximate values just found in three terms of that order which are given in (92), and take $a_1^2 = 3{\cdot}7$, we now obtain $-k^2a_1^6 = 2027$, 1/10 of the increase being due to the last of the three terms. With this value we finally obtain

$$D\overline{W}\rho/\mu = C'a^3\rho/\mu = -4ika_1^3 = 180.$$

It appears that we may safely neglect terms in which higher powers of k than the fourth occur; the term involving k^6 which is given in (92) is presumably the most important of these; and on substitution of the numbers just found, its value is seen to be about 1/1000.