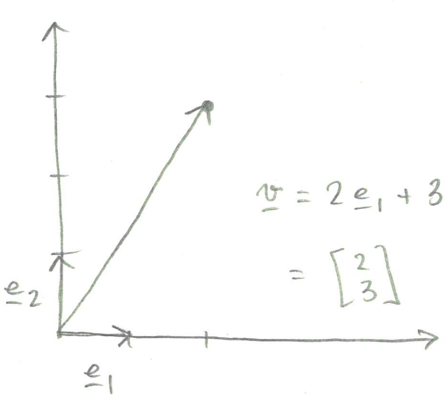
Lecture I: Review of Vector & Tensor Algebra

I. What is a vector and tensor?

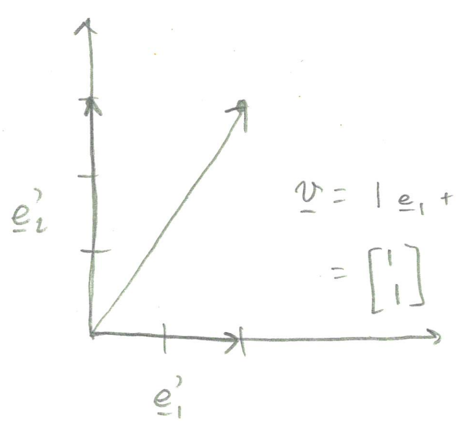
A. Abstract vector spaces

* A vector space is an abstract mathematical set that satisfies certain axioms (closed under addition: $\underline{a} + \underline{b} = \underline{c}$; closed under scalar multiplication: $c\,\underline{a} = \underline{b}$; etc.) We'll leave these details to the math department. You know many examples:
    - $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \ldots$ n-tuples of real numbers
    - set of all $m \times n$ matrices
    - set of all continuous, real functions on $[a,b]$
* We need two things to describe an "abstract vector" or an object in a vector space:
    - A basis (a "coordinate system")
    - A set of coordinates/components (an "address" in the basis)

* *Example*: vectors in $\mathbb{R}^2$



$\underline{v} = 2\underline{e}_1 + 3\underline{e}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$



$\underline{v} = 1\underline{e}_1 + 1\underline{e}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

* Some comments:

  - The basis on the left ($\underline{e}_1, \underline{e}_2$) is the "natural basis" for $\mathbb{R}^2$. It is orthonormal: $\underline{e}_1 \cdot \underline{e}_2 = 0$; $|\underline{e}_1| = |\underline{e}_2| = 1$

  - The same vector is shown in both plots with a different basis & different coordinates

  - There are many different valid bases. A basis must (1) have linearly independent basis vectors and (2) span the vector space. More on this later.

B. Scalars, vectors, & Tensors

* We'll talk more about abstract vector spaces in a moment. In transport we will care about objects that need:

	name	example
• 0 basis vectors	scalar	5
• 1 basis vector	(geometric) vector	$3\underline{e}_2$
• 2 basis vectors	tensor	$7\underline{e}_1\underline{e}_2$

* Scalars & (geometric) vectors should be familiar. For example, a vector in $\mathbb{R}_3$ is:

$$\underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3 = \sum_{i=1}^{3} v_i \underline{e}_i = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

* Tensors are perhaps less familiar. They are defined using a special product of two basis vectors in $\mathbb{R}^3$. This is called a direct product or a dyadic product.

$\underline{e}_1 \underline{e}_2$ alt. notation $\underline{e}_1 \otimes \underline{e}_2$

* These unit dyads $\underline{e}_i \underline{e}_j$ form a basis in the tensor vector space that consist of two directions.

* Just like with a (geometric) vector, a tensor is specified by a basis and a set of coordinates:

$$
\begin{aligned}
\underline{\underline{T}} &= T_{11}\, \underline{e}_1 \underline{e}_1 + T_{12}\, \underline{e}_1 \underline{e}_2 + T_{13}\, \underline{e}_1 \underline{e}_3 \\
&\quad + T_{21}\, \underline{e}_2 \underline{e}_1 + T_{22}\, \underline{e}_2 \underline{e}_2 + T_{23}\, \underline{e}_2 \underline{e}_3 \\
&\quad + T_{31}\, \underline{e}_3 \underline{e}_1 + T_{32}\, \underline{e}_3 \underline{e}_2 + T_{33}\, \underline{e}_3 \underline{e}_3 \\
&= \sum_{i=1}^{3} \sum_{j=1}^{3} T_{ij}\, \underline{e}_i \underline{e}_j = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}
\end{aligned}
$$

* comments:

  - we need a 3x3 matrix for a (rank-2) tensor!
  - By convention in T_{ij} i are the rows & j are the columns.
  - If we have a triad $(\underline{e}_i \underline{e}_j \underline{e}_k)$ or a tetrad $(\underline{e}_i \underline{e}_j \underline{e}_k \underline{e}_l)$ we can have rank-3 or rank-4 tensors. Scalars and (geometric) vectors are rank-0 & rank-1 tensors.

C. Notation

* There are two primary sets of notation for vectors / tensors:

  - Gibbs Notation : $\underline{v}$, $\underline{\underline{A}}$, $\underline{\nabla}$ ← bold in book

    - common, compact, pretty

- Index Notation: v_i, A_{ij}, ∂_i

  - Deen calls this "cartesian tensor" notation
  - Good for proofs, computers
  - AKA "Einstein" notation

* Index notation is a short-hand for full component notation:

$$\underbrace{\underline{v}}_{\substack{\text{Gibbs}\\ \text{notation}}} = \underbrace{\sum_i v_i \underline{e}_i}_{\substack{\text{component notation}\\ \text{w/ basis vectors}}} = \underbrace{v_i}_{\substack{\text{shorthand, sum}\\ \text{\& basis implied}}}$$

* It is also useful to represent vectors & tensors with bracket / matrix notation:

$$\underline{v} = v_i = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \text{column vector}$$

$$\underline{w} = w_i = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \quad \text{row vector}$$

$$\underline{\underline{T}} = T_{ij} = \begin{bmatrix} T_{xx} & T_{xy} \\ T_{yx} & T_{yy} \end{bmatrix} \quad \text{matrix}$$

II. Vector & Tensor Properties and Algebraic Operations

* I'm going to review some select properties and operations: we don't have time to be comprehensive. Please read Appendix A in Deen for more details. You will be responsible for all the material contained therein.

A. List of operations in Appendix

* Addition
* subtraction
* Transpose
* Decomposition into symmetric & anti-symmetric parts
* Scalar multiplication / division
* dot product of vectors
* cross product of vectors
* multiple products of vectors
* dyadic product
* tensor dot product
* tensor double-dot product
* magnitude of a tensor

B. List of properties

* symmetric / anti-symmetric tensor
* products: commutative, associative, distributive

C. Special tensors

* Kronecker delta / Identity tensor, δ_{ij} | $\underset{=}{\delta}$
* permutation symbol / alte

D. Examples

* Dot product: $\underset{\sim}{e}_1 \cdot$

(I do first example) $\underset{\sim}{e}_1 \cdot \underset{=}{A} \cdot \underset{\sim}{e}_3 =$

I liked what I did here. I had students look in the appendix for a product / operation that was unfamiliar. Then I did an example. Next year, prep one on symmetric/anti-symmetric.

$$= \sum_i \sum_j A_{ij} \; \underline{e}_1 \cdot \underline{e}_i \; \underline{e}_j \cdot \underline{e}_3$$

on page 613

$$\underline{e}_i \cdot \underline{e}_j = \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

kronecker delta

$$\underline{e}_1 \cdot \underline{e}_i = 1 \quad \text{when } i=1$$
$$= 0 \quad \text{otherwise}$$

$$e_j \cdot \underline{e}_3 = 1 \quad \text{when } j=3$$
$$= 0 \quad \text{otherwise}$$

$$= A_{11} \cdot 1 \cdot 0 + A_{12} \cdot 1 \cdot 0 + A_{13} \cdot 1 \cdot 1$$
$$+ A_{21} \cdot 0 \cdot 0 + A_{22} \cdot 0 \cdot 0 + A_{23} \cdot 0 \cdot 1$$
$$+ A_{31} \cdot 0 \cdot 0 + A_{32} \cdot 0 \cdot 0 + A_{33} \cdot 0 \cdot 1$$

$$= A_{13}$$

(class does examples 2 & 3)

* Now you try a Dyadic Product: $\underline{v}\underline{w}$ (page 615)

$$\underline{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \underline{w} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\underline{v}\underline{w} = \sum_i \sum_j v_i w_j \; \underline{e}_i \underline{e}_j$$

$$= v_1 w_1 \underline{e}_1 \underline{e}_1 + v_1 w_2 \underline{e}_1 \underline{e}_2 + v_1 w_3 \underline{e}_1 \underline{e}_3$$
$$+ v_2 w_1 \underline{e}_2 \underline{e}_1 + v_2 w_2 \underline{e}_2 \underline{e}_2 + v_2 w_3 \underline{e}_2 \underline{e}_3$$
$$+ v_3 w_1 \underline{e}_3 \underline{e}_1 + v_3 w_2 \underline{e}_3 \underline{e}_2 + v_3 w_3 \underline{e}_3 \underline{e}_3$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 4 & 0 & 2 \\ 6 & 0 & 3 \end{bmatrix}$$

* (If time or other half of the class)

A double dot product: $\underline{\underline{A}} : \underline{\underline{B}}$

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \underline{\underline{B}} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$\underline{\underline{A}} : \underline{\underline{B}} = \left(\sum_i \sum_j A_{ij} \, \underline{e}_i \underline{e}_j\right) : \left(\sum_k \sum_l B_{kl} \, \underline{e}_k \underline{e}_l\right)$$

$$= \sum_i \sum_j \sum_k \sum_l A_{ij} B_{kl} \; \underline{e}_i \underline{e}_j : \underline{e}_k \underline{e}_l$$

↑ like 2 dot products

$$= \sum_i \sum_j \sum_k \sum_l A_{ij} B_{kl} \, \delta_{jk} \delta_{il} \left\{ \begin{array}{l} \underline{e}_j \cdot \underline{e}_k = \delta_{jk} \\ \underline{e}_i \cdot \underline{e}_l = \delta_{il} \end{array} \right.$$

$$= \sum_i \sum_j A_{ij} B_{ji}$$

$$= A_{11} B_{11} + A_{12} B_{21} + A_{21} B_{12} + A_{22} B_{22}$$

$$= 1 \cdot 2 + 2 \cdot (-1) + 3(1) + 3(4)$$

$$= 2 - 2 + 3 + 12 = 15$$