Lecture 2 - Coordinate systems

I. Introduction to coordinate systems

* Today we are going to examine coordinate systems more closely

* A coordinate system is a choice of basis vectors in the vector space, i.e. $\mathbb{R}^3$.

* For example, the choice:

$$\underset{\sim}{e}_1 = \underset{\sim}{e}_x \,, \quad \underset{\sim}{e}_2 = \underset{\sim}{e}_y \,, \quad \underset{\sim}{e}_3 = \underset{\sim}{e}_z$$

is the familiar Cartesian coordinate system. It is the "natural" basis set for $\mathbb{R}^3$.

* Why choose a different c.s.? We are representing the same vectors after all. Is this just torture?

  (1) mathematical convenience - The natural geometry of the problem may lend itself to a particular coordinate system, e.g. flow around a sphere.

  (2) Understanding - We may be able to solve our problem in multiple coordinate systems, but one may provide especially clear insight. For example, which is more easily recognized as a circle: $x^2+y^2=R^2$ or $r=R$?

A. Examples of coordinate systems

* let's look at some examples. At the same time, it is useful to examine some properties of these C.S.'s that we can use to classify them.

* Properties:

(1) Linear - There is a 1-to-1 mapping between the C.S. & the Cartesian C.S. All examples here are linear. See differential geometry / general relativity for exceptions.

(2) Homogeneous - The basis vectors are constants throughout the domain

(3) Curvilinear - The basis vectors are functions of space. opposite of homogeneous.

(4) Orthogonal - The basis vectors obey the relation $\underline{e}_i \cdot \underline{e}_j = 0$ when $i \neq j$.

"orthonormal"

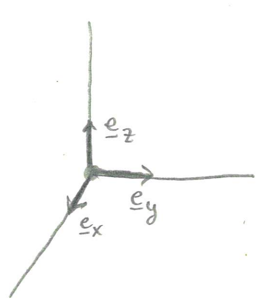
(5) Skew - opposite of orthogonal: $\underline{e}_i \cdot \underline{e}_j \neq 0$ when $i \neq j$. Not very common outside of relativity.

(6) Normalized - Basis vectors are unit vectors: $\underline{e}_i \cdot \underline{e}_j = 1$

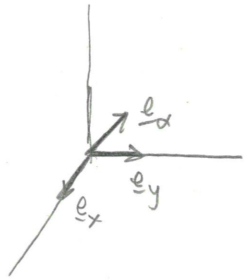
(7) Right-handed - C.S. follows the right-hand rule: $\underline{e}_1 \times \underline{e}_2 = \underline{e}_3$

(8) left-handed - C.S. follows the left-hand rule: $\underline{e}_1 \times \underline{e}_2 = -\underline{e}_3$

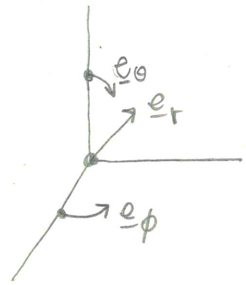
* Examples:



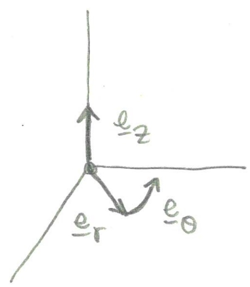
Cartesian: homogeneous, orthonormal
(x,y,z) right-handed.



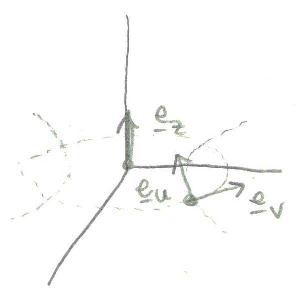
<u>oblique</u> : homogeneous, skew, normalized,
(x, y, α) right-handed



<u>spherical</u> : curvilinear, orthonormal,
(r, θ, ϕ) right-handed.



<u>cylindrical</u> : curvilinear, orthonormal,
(r, θ, z) right-handed.



<u>elliptic</u> : curvilinear, orthogonal (not normal),
(u, v, z) right-handed (I think)

B. Mathematical Definition of a coordinate system

* Now, let's be more specific about the math. As we've said, we need to define <u>basis vectors</u>.

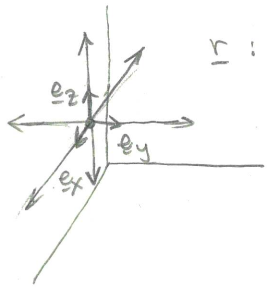
* How do we define these vectors?

(1) Define a function w/ the desired geometry ($\underline{r}$)

(2) Find tangents to the curve. (derivatives)

(3) Normalize the tangent vectors

* Cartesian coordinates



$\underline{r}$: straight lines.

$$\underline{r} = x\,\underline{e}_x + y\,\underline{e}_y + z\,\underline{e}_z$$

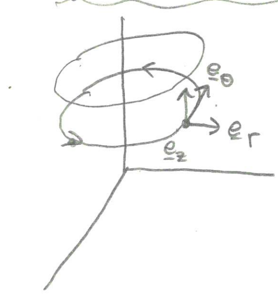
a vector function that encodes our geometry.

- tangents are easy: $\underline{e}_x, \underline{e}_y, \underline{e}_z$
- we know that these are orthonormal:

$$\left.\begin{array}{l} \underline{e}_x \cdot \underline{e}_y = 0 \\ \underline{e}_x \cdot \underline{e}_x = 1 \end{array}\right\}$$

$$\underline{e}_i \cdot \underline{e}_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

* Cylindrical coordinates



$\underline{r}$: cylinder shape

$$\underline{r} = r\cos\theta\,\underline{e}_x + r\sin\theta\,\underline{e}_y + z\,\underline{e}_z$$

$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$

[⊛ Note: $|\underline{r}| \neq r$. Following book notation here.]

- How do we get tangents?

tangent to θ : $\dfrac{\partial \underline{r}}{\partial \theta}$

Let's do it.

$$\frac{\partial \underline{r}}{\partial \theta} = -r\sin\theta\,\underline{e}_x + r\cos\theta\,\underline{e}_y + 0\,\underline{e}_z$$

- Are they unit normals?

$$\left|\frac{\partial \underline{r}}{\partial \theta}\right| = \sqrt{r^2\sin^2\theta + r^2\cos^2\theta} = r\sqrt{\sin^2\theta + \cos^2\theta}$$

$$= r$$

- not normalized, but that is easy:

$$\underline{e}_\theta = \frac{1}{\left|\frac{\partial \underline{r}}{\partial \theta}\right|}\frac{\partial \underline{r}}{\partial \theta} = \frac{1}{r}\left(-r\sin\theta\,\underline{e}_x + r\cos\theta\,\underline{e}_y\right)$$

$$\boxed{\underline{e}_\theta = -\sin\theta\,\underline{e}_x + \cos\theta\,\underline{e}_y}$$

- Repeat for $\underline{e}_r$ & $\underline{e}_z$

$$\boxed{\begin{aligned} \underline{e}_r &= \cos\theta\,\underline{e}_x + \sin\theta\,\underline{e}_y \\ \underline{e}_z &= \underline{e}_z \end{aligned}}$$

- Comment: notice the basis vectors are <u>functions of θ</u>. This makes them non-homogeneous.

- Are they orthogonal? (Yes!)

C. General Mathematical Formulas

* we can generalize beyond these cases to give a robust procedure for deriving these unit vectors.

(1) Define a vector function $\underline{r}$, that encodes the geometry.

$$\underline{r} = x(u_1,u_2,u_3)\,\underline{e}_x + y(u_1,u_2,u_3)\,\underline{e}_y + z(u_1,u_2,u_3)\,\underline{e}_z$$

- u's are like r, θ, z. variables for new geometry.
- This defines a mapping between $xyz \leftrightarrow u_1 u_2 u_3$.

(2) Find the tangent vectors to this function.

$$\underline{a}_i = \frac{\partial \underline{r}}{\partial u_i}$$

(3) Normalize the tangent vector.

$$h_i = \left| \frac{\partial \underline{r}}{\partial u_i} \right| = \left| \underline{a}_i \right| \quad \leftarrow \text{"scale factor"}$$

$$\underline{e}_i = \frac{1}{h_i} \underline{a}_i \quad \leftarrow \text{normalized unit vector}$$

* Comments:

- The scale factor, h_i, will be a useful thing that will come up again.

- One can also find $u_1(x, y, z)$ and $\underline{e}_x = f(\underline{e}_{u_1}, \underline{e}_{u_2}, \underline{e}_{u_3})$ by inverting the relationships we derived. If the system is linear, then they are related by

$$\begin{bmatrix} \underline{e}_{u_1} \\ \underline{e}_{u_2} \\ \underline{e}_{u_3} \end{bmatrix} = \underbrace{\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}}_{\underline{\underline{Q}}} \begin{bmatrix} \underline{e}_x \\ \underline{e}_y \\ \underline{e}_z \end{bmatrix}$$

  All these systems are linear. If the c.s. is orthonormal, the inverting is easy: $\underline{\underline{Q}}^{-1} = \underline{\underline{Q}}^t$

  See Deen A.7-30 & A.7-31 for examples.

- Spherical coordinates in HW:

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

Vector Products in curvilinear Coordinates

what is $\underline{v} \cdot \underline{w}$ in curvilinear coordinates?

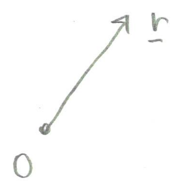
* The dot product is given by

$$\underline{v} \cdot \underline{w} = \sum_i v_i w_i$$

for any orthogonal coordinate system. A similar result holds for other products.

* Be careful. This does not apply to products of position vectors. Position vectors are given by:

$$\underline{r} = x\, \underline{e}_x + y\, \underline{e}_y + z\, \underline{e}_z$$



Example 1: Regular dot product of field variables

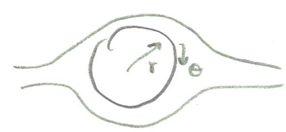
Consider the velocity field in cylindrical coordinates

$$\underline{v} = v_r\, \underline{e}_r + v_\theta\, \underline{e}_\theta$$

$$v_r = u \cos\theta \left[1 - \left(\frac{R}{r}\right)^2 \right]$$

$$v_\theta = -u \sin\theta \left[1 + \left(\frac{R}{r}\right)^2 \right]$$

Flow at high Re around a cylinder



(a) In cylindrical coordinates

$$\underline{v} \cdot \underline{v} = v_r^2 + v_\theta^2$$

$$v_r^2 = u^2 \cos^2\theta \left[1 - \left(\frac{R}{r}\right)\right]^2$$

$$v_\theta^2 = u^2 \sin^2\theta \left[1 + (R/r)^2\right]^2$$

$$v^2 = v_r^2 + v_\theta^2 = u^2 \cos^2\theta \left[1 - 2\left(\frac{R}{r}\right)^2 + \left(\frac{R}{r}\right)^4\right]$$

$$+ u^2 \sin^2\theta \left[1 + 2\left(\frac{R}{r}\right)^2 + \left(\frac{R}{r}\right)^4\right]$$

$$= u^2 \left(\cos^2\theta + \sin^2\theta\right)\left[1 + (R/r)^4\right]$$

$$+ u^2 \left(\sin^2\theta - \cos^2\theta\right)(2)(R/r)^2$$

recall: $\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta - \sin^2\theta = \cos 2\theta$

$$\boxed{\frac{v^2}{u^2} = 1 + 2\left(\frac{R}{r}\right)^2 \cos(2\theta) + \left(\frac{R}{r}\right)^4}$$

(b) In cartesian coordinates

* Convert to cartesian coordinates first

$$\left.\begin{aligned} \underline{e}_r &= \cos\theta\, \underline{e}_x + \sin\theta\, \underline{e}_y \\ \underline{e}_\theta &= -\sin\theta\, \underline{e}_x + \cos\theta\, \underline{e}_y \end{aligned}\right\} \text{relation between unit vectors}$$

$$\underline{v} = v_x \underline{e}_x + v_y \underline{e}_y = \underbrace{v_r \underline{e}_r + v_\theta \underline{e}_\theta}_{\text{write this and sub in } \underline{e}_r \,\&\, \underline{e}_\theta}$$

$$= u\cos\theta\left[1 - \left(\frac{R}{r}\right)^2\right] \times \left(\cos\theta\, \underline{e}_x + \sin\theta\, \underline{e}_y\right)$$

$$- u\sin\theta\left[1 + \left(\frac{R}{r}\right)^2\right]\left(-\sin\theta\, \underline{e}_x + \cos\theta\, \underline{e}_y\right)$$

$$= u\cos^2\theta\left[1 - (R/r)^2\right]\underline{e}_x + u\cos\theta\sin\theta\left[1 - (R/r)^2\right]\underline{e}_y$$

$$+ u\sin^2\theta\left[1 + (R/r)^2\right]\underline{e}_x - u\cos\theta\sin\theta\left[1 + (R/r)^2\right]\underline{e}_y$$

$$v_x = u\cos^2\theta\left[1 - (R/r)^2\right] + u\sin^2\theta\left[1 + (R/r)^2\right]$$

$$v_y = u\cos\theta\sin\theta\,\{\cancel{1} - (R/r)^2 - \cancel{1} - (R/r)^2]$$

$\Downarrow$ simplify both

$$v_x = u\cos^2\theta + u\sin^2\theta - u\cos^2\theta\,(R/r)^2 + u\sin^2\theta\,(R/r)^2$$

$$= u[\underbrace{\cos^2\theta + \sin^2\theta}_{1}] - u\left(\frac{R}{r}\right)^2[\underbrace{\cos^2\theta - \sin^2\theta}_{\cos 2\theta}]$$

$$v_x = u - u\left(\frac{R}{r}\right)^2\cos 2\theta \quad \checkmark$$

$$v_y = -2u\cos\theta\sin\theta\left(\frac{R}{r}\right)^2$$

Now, compute: $\underline{v}\cdot\underline{v}$

$$v^2 = \underline{v}\cdot\underline{v} = v_x^2 + v_y^2 = u^2\left[1 - \left(\frac{R}{r}\right)^2\cos 2\theta\right]^2 + 4u^2\cos^2\theta\sin^2\theta\,(R/r)^4$$

$$= u^2\left[1 - 2\left(\frac{R}{r}\right)^2\cos 2\theta + \left(\frac{R}{r}\right)^4\cos^2 2\theta\right] + 4u^2\cos^2\theta\sin^2\theta\,(R/r)^4$$

$$\frac{v^2}{u^2} = 1 - 2\left(\frac{R}{r}\right)^2\cos 2\theta + \left(\frac{R}{r}\right)^4\left[\cos^2 2\theta + 4\cos^2\theta\sin^2\theta\right]$$

expand & simplify

$$\cos^2(2\theta) = (\cos^2\theta - \sin^2\theta)^2 = \cos^2\theta\cos^2\theta - 2\cos^2\theta\sin^2\theta + \sin^2\theta\sin^2\theta$$

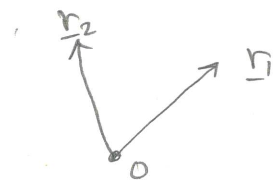
$$\cos^2(2\theta) + 4\cos^2\theta\sin^2\theta = \cos^2\theta\cos^2\theta + 2\cos^2\theta\sin^2\theta + \sin^2\theta\sin^2\theta$$

$$= (\cos^2\theta + \sin^2\theta)^2 = 1^2 = 1$$

$$\boxed{\frac{v^2}{u^2} = 1 - 2(R/r)^2\cos 2\theta + (R/r)^4}$$

Same! The formula is correct.

Example 2: Dot product of position vectors



$$\underline{r}_1 = x_1 \underline{e}_x + y_1 \underline{e}_y + z_1 \underline{e}_z$$
$$\underline{r}_2 = x_2 \underline{e}_x + y_2 \underline{e}_y + z_2 \underline{e}_z$$

Dot product: $\underline{r}_1 \cdot \underline{r}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2$

* Convert to cylindrical coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

$$\underline{r}_1 \cdot \underline{r}_2 = r_1\cos\theta_1 \cdot r_2\cos\theta_2 + r_1\sin\theta_1 \cdot r_2 \sin\theta_2 + z_1 z_2$$
$$= r_1 r_2 \underbrace{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)}_{\cos(\theta_1-\theta_2)} + z_1 z_2 \quad \leftarrow \text{trig identity}$$

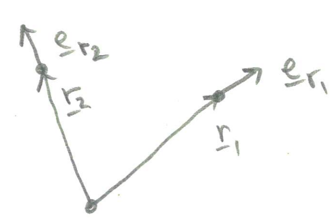
$$\boxed{\underline{r}_1 \cdot \underline{r}_2 = r_1 r_2 \cos(\theta_1 - \theta_2) + z_1 z_2}$$

* Compare this to the formula valid for field variables

$$\underline{r}_1 = r_1 \underline{e}_r + z_1 \underline{e}_z \qquad \underline{r}_2 = r_2 \underline{e}_r + z_2 \underline{e}_z$$

$$\boxed{\underline{r}_1 \cdot \underline{r}_2 = r_1 r_2 + z_1 z_2} \quad \times \text{ wrong!}$$

why? Because $\underline{e}_r$ for $\underline{r}_1$ is not the same as $\underline{e}_r$ for $\underline{r}_2$:



$$\underline{e}_r = \cos\theta\, \underline{e}_x + \sin\theta\, \underline{e}_y$$

* Let's try this instead!

$$\underline{r}_1 \cdot \underline{r}_2 = (r_1\, \underline{e}_{r_1}) \cdot (r_2\, \underline{e}_{r_2}) + z_1\, \underline{e}_{z_1} \cdot z_2\, \underline{e}_{z_2}$$

$$= r_1 r_2 \underbrace{(\underline{e}_{r_1} \cdot \underline{e}_{r_2})} + (z_1 z_2) \underbrace{(\underline{e}_{z_1} \cdot \underline{e}_{z_2})}_{1}$$

$$\underline{e}_{r_1} \cdot \underline{e}_{r_2} = (\cos\theta_1\, \underline{e}_x + \sin\theta_1\, \underline{e}_y) \cdot (\cos\theta_2\, \underline{e}_x + \sin\theta_2\, \underline{e}_y)$$

$$= \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$$

using $\underline{e}_x \cdot \underline{e}_x = 1$, $\underline{e}_y \cdot \underline{e}_y = 1$

$\underline{e}_x \cdot \underline{e}_y = 0$

$$= \cos(\theta_1 - \theta_2) \quad \text{(trig identity)}$$

$$\boxed{\underline{r}_1 \cdot \underline{r}_2 = r_1 r_2 \cos(\theta_1 - \theta_2) + z_1 z_2}$$

This is correct!

* Moral of the Story: when in doubt, use cartesian coordinates for position vectors, be very careful! They are not the same if they are at different points in space.