Lecture 3: Vector & Tensor Calculus

* Now that we have a deeper view of coordinate systems, we can dig into vector/tensor calculus.
* Check out supplemental notes for info about vector algebra in curvilinear coordinates.

I. Differential Calculus

* Derivative operations with vectors require the $\underline{\nabla}$ operator (gradient, del, nabla).

A. Overview of operations

* The gradient operator is a vector operator. It is defined as:

$$\underline{\nabla} \equiv \sum_i \frac{1}{h_i} \underline{e}_i \frac{\partial}{\partial u_i} = \partial_i$$

shorthand index notation

scale factor

coordinate

* Looks similar to when we got unit vectors: $\xrightarrow{u_i} \frac{\partial}{\partial u_i}$, normalize w/ h_i

* Aside: Note the dimensional consistency. Good check

* example: Cartesian

$u_i = \{x, y, z\}$ $h_x = h_y = h_z = 1$

$$\underline{\nabla} = \underline{e}_x \frac{\partial}{\partial x} + \underline{e}_y \frac{\partial}{\partial y} + \underline{e}_z \frac{\partial}{\partial z}$$

* example: cylindrical

$u_i = \{r, \theta, z\}$ $h_r = 1$, $h_\theta = r$, $h_z = 1$

$$\underline{\nabla} = \underline{e}_r \frac{\partial}{\partial r} + \frac{1}{r} \underline{e}_\theta \frac{\partial}{\partial \theta} + \underline{e}_z \frac{\partial}{\partial z}$$

* There are four common differential operations:

- Gradient: $\underline{\nabla} s$, $\underline{\nabla}\, \underline{v}$
  - Direct product (dyadic product)
  - "Adds" one dimension: scalar → vector
    vector → tensor

- Divergence: $\underline{\nabla} \cdot \underline{v}$, $\underline{\nabla} \cdot \underline{\underline{\tau}}$
  - Inner/Dot product
  - "subtracts" one dimension: vector → scalar
    tensor → vector

- Curl: $\underline{\nabla} \times \underline{v}$
  - Cross product
  - For vectors only: vector → vector

- Laplacian: $\nabla^2 s$, $\nabla^2 \underline{v}$, $\nabla^2 \underline{\underline{\tau}}$
  - Scalar operation: $\underline{\nabla} \cdot \underline{\nabla} = \nabla^2$
  - "2nd derivative", for all orders
  - Not the same thing as $\underline{\nabla}\,\underline{\nabla}$ ← dyadic of two gradients.

B. Examples

* We'll stick to cylindrical coordinates. Shows details, but not as messy as spherical (on the HW).

* How derive expressions for derivative operations?

  (1) Use definitions. "Plug & chug" derivations. Be systematic

  (2) Remember: The unit vectors can be functions of space!!

* Example 1:

what is $\underline{\nabla} \cdot \underline{v}$ in cylindrical coordinates?

$$u_i = \{r, \theta, z\} \qquad h_r = 1, \; h_\theta = r, \; h_z = 1$$

$$\underline{\nabla} \cdot \underline{v} = \underbrace{\left[\sum_i \frac{1}{h_i} \underline{e}_i \frac{\partial}{\partial u_i} \right]}_{\underline{\nabla}} \cdot \underbrace{\left[\sum_i v_i \underline{e}_i \right]}_{\underline{v}}$$

$$= \left[\underline{e}_r \frac{\partial}{\partial r} + \frac{1}{r} \underline{e}_\theta \frac{\partial}{\partial \theta} + \underline{e}_z \frac{\partial}{\partial z} \right] \cdot \left[v_r \underline{e}_r + v_\theta \underline{e}_\theta + v_z \underline{e}_z \right]$$

$$= \underline{e}_r \frac{\partial}{\partial r} \cdot \left(v_r \underline{e}_r + v_\theta \underline{e}_\theta + v_z \underline{e}_z \right) + \frac{1}{r} \underline{e}_\theta \frac{\partial}{\partial \theta} \cdot \left(v_r \underline{e}_r + v_\theta \underline{e}_\theta + v_z \underline{e}_z \right) + \underline{e}_z \frac{\partial}{\partial z} \cdot \left(v_r \underline{e}_r + v_\theta \underline{e}_\theta + v_z \underline{e}_z \right)$$

- swap order of $\cdot$ & $\frac{\partial}{\partial u_i}$
- The $\underline{e}_i$ are not constants (in general)

$$\frac{\partial}{\partial \theta}(\underline{e}_r) = \underline{e}_\theta \qquad \frac{\partial}{\partial \theta}(\underline{e}_\theta) = -\underline{e}_r \quad (*)$$

rest are zero (they are constants)

(*) Proof ommitted, see §A.7, pp 629 - 633 for unit vectors & derivatives.

- If $\frac{\partial}{\partial u_i} \underline{e}_j \neq 0$, then need the product rule.
- Go term by term:

$$\underline{e}_r \cdot \frac{\partial}{\partial r}(v_r \underline{e}_r) = \underline{e}_r \cdot \underline{e}_r \frac{\partial v_r}{\partial r}$$

$$\underline{e}_r \cdot \frac{\partial}{\partial r}(v_\theta \underline{e}_\theta) = \underline{e}_r \cdot \underline{e}_\theta \frac{\partial v_\theta}{\partial r}$$

$$\underline{e}_r \cdot \frac{\partial}{\partial r}(v_z \underline{e}_z) = \underline{e}_r \cdot \underline{e}_z \frac{\partial v_z}{\partial r}$$

$$\underline{e}_\theta \cdot \frac{\partial}{\partial \theta}(v_r \underline{e}_r) = \underline{e}_\theta \cdot \underline{e}_r \frac{\partial v_r}{\partial \theta} + \underline{e}_\theta \cdot \left(v_r \frac{\partial \underline{e}_r}{\partial \theta} \right)$$

$$= \underline{e}_\theta \cdot \underline{e}_r \frac{\partial v_r}{\partial \theta} + \underline{e}_\theta \cdot \underline{e}_\theta \, v_r$$

$$\underline{e}_\theta \cdot \frac{\partial}{\partial \theta}(v_\theta \underline{e}_\theta) = \underline{e}_\theta \cdot \underline{e}_\theta \frac{\partial v_\theta}{\partial \theta} + \underline{e}_\theta \cdot \left(v_\theta \frac{\partial \underline{e}_\theta}{\partial \theta} \right)$$

$$= \underline{e}_\theta \cdot \underline{e}_\theta \frac{\partial v_\theta}{\partial \theta} + \underline{e}_\theta \cdot (-\underline{e}_r) \, v_\theta$$

$$\underline{e}_\theta \cdot \frac{\partial}{\partial \theta}(v_z \underline{e}_z) = \underline{e}_\theta \cdot \underline{e}_z \frac{\partial v_z}{\partial \theta}$$

$$\underline{e}_z \cdot \frac{\partial}{\partial z}(v_r \underline{e}_r) = \underline{e}_z \cdot \underline{e}_r \frac{\partial v_r}{\partial z}$$

$$\underline{e}_z \cdot \frac{\partial}{\partial z}(v_\theta \underline{e}_\theta) = \underline{e}_z \cdot \underline{e}_\theta \frac{\partial v_\theta}{\partial z}$$

$$\underline{e}_z \cdot \frac{\partial}{\partial z}(v_z \underline{e}_z) = \underline{e}_z \cdot \underline{e}_z \frac{\partial v_z}{\partial z}$$

- The base vectors are orthonormal:

$$\underline{e}_i \cdot \underline{e}_j = \delta_{ij} \longrightarrow \quad \underline{e}_r \cdot \underline{e}_\theta = 0$$
$$\underline{e}_r \cdot \underline{e}_r = 1$$

$$\boxed{\nabla \cdot \underline{v} = \frac{\partial v_r}{\partial r} + \frac{1}{r} v_r + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}}$$

- These are all given in Table A-2, A-3, A-4 in Deen.

* How evaluate these derivative expressions?

* Example 2: what is $\nabla \underline{v}$ for $\underline{v} = ar - \frac{b}{r} \underline{e}_\theta$

- table A-3 in Deen (p. 632)
- 9 components
- $\partial_i v_j$

  row (i), column (j)

This example uses the table. Do this faster. Don't derive from scratch.

$$\underline{\nabla}\,\underline{v} = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{\partial v_\theta}{\partial r} & \frac{\partial v_z}{\partial r} \\ \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} & \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} & \frac{1}{r}\frac{\partial v_z}{\partial \theta} \\ \frac{\partial v_r}{\partial z} & \frac{\partial v_\theta}{\partial z} & \frac{\partial v_z}{\partial z} \end{bmatrix}$$

$$v_r = 0 \quad , \quad v_\theta = ar - \frac{b}{r} \quad , \quad v_z = 0$$

$\frac{\partial v_\theta}{\partial r} \neq 0$, all else 0

$$\underline{\nabla}\,\underline{v} = \begin{bmatrix} 0 & a + \frac{b}{r^2} & 0 \\ -a + \frac{b}{r^2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

* How derive vector differential identities? (Table A-1)

- Use cartesian notation. An identity is independent of coordinate system

* Example 3: what is $\underline{\nabla} \cdot (f\underline{a})$

$$\underline{\nabla} \cdot (f\underline{a}) = \left(\sum_i \frac{1}{h_i} \underline{e}_i \frac{\partial}{\partial u_i}\right) \cdot \left(f \sum_j a_j \underline{e}_j\right)$$

assume cartesian

$h_i = 1$, $\underline{e}_i$ are constants

$$= \left(\sum_i \underline{e}_i \frac{\partial}{\partial u_i}\right) \cdot \left(f \sum_j a_j \underline{e}_j\right)$$

"FOIL"

$$= \sum_i \sum_j \underline{e}_i \cdot \underline{e}_j \frac{\partial}{\partial u_i}(f a_j)$$

product rule

$$= \sum_i \sum_j \underline{e}_i \cdot \underline{e}_j \left[f \frac{\partial a_j}{\partial u_i} + a_j \frac{\partial f}{\partial u_i} \right]$$

$$= \sum_i \sum_j \left(f\, \underline{e}_i \frac{\partial}{\partial u_i}\right) \cdot (a_j\, \underline{e}_j)$$

$$+ \sum_i \sum_j \left(\underline{e}_i \frac{\partial f}{\partial u_i}\right) \cdot (\underline{e}_j\, a_j)$$

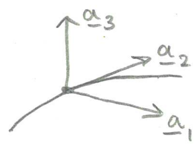
$$= f\, \underline{\nabla} \cdot \underline{a} + \underline{\nabla} f \cdot \underline{a}$$

$$\boxed{\underline{\nabla} \cdot (f\underline{a}) = f(\underline{\nabla} \cdot a) + \underline{\nabla} f \cdot \underline{a}}$$

II. Integral Calculus

A. Volume & surface Integrals

* volume & surface integrals have differential elements that depend on space



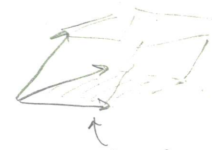
$$a_i = h_i\, \underline{e}_i$$

$$dV = \underline{a}_1 \cdot (\underline{a}_2 \times \underline{a}_3)$$

volume of parallelpiped

$$dS_i = \underline{a}_j \times \underline{a}_k$$

surface $\perp$ to j & k



* plugging into these geometric expressions gives

$$dV = h_1 h_2 h_3\, du_1\, du_2\, du_3$$

$$dS_1 = h_2 h_3\, du_2\, du_3$$

$$dS_2 = h_1 h_3\, du_1\, du_3$$

$$dS_3 = h_1 h_2\, du_1\, du_2$$

* <u>Example:</u> cylindrical coordinates

$$u_i = \{r, \theta, z\}, \quad h_r = 1, \; h_\theta = r, \; h_z = 1$$

$dV = r\,dr\,d\theta\,dz$

$dS_r = r\,d\theta\,dz$

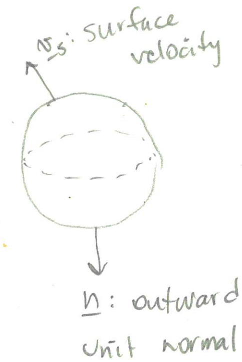
$dS_\theta = dr\,dz$

$dS_z = r\,dr\,d\theta$

B. Integral Identities

* There are two important integral identities

* "Gauss Divergence Theorem



scalar $$\int_V \nabla f\,dV = \int_S \underline{n} f\,dS$$

vector $$\int_V \underline{\nabla}\cdot\underline{v}\,dV = \int_S \underline{n}\cdot\underline{v}\,dS$$

tensor $$\int_V \underline{\nabla}\cdot\underline{\underline{T}}\,dV = \int_S \underline{n}\cdot\underline{\underline{T}}\,dS$$

(*) I expect you to be able to derive everything in the appendix. However, after this HW, just use the tables (except on Exam if asked).

- Relationship between volume & surface integrals
- Useful for deriving balances
- "Fundamental Theorem of Calculus" for 2D/3D

* Leibniz Rule

1D $$\frac{d}{dt}\int_{A(t)}^{B(t)} f(x,t)\,dx = \int_{A(t)}^{B(t)} \frac{\partial f}{\partial t}\,dx + \frac{dB}{dt} f[B(t),t] - \frac{dA}{dt} f[A(t),t]$$

3D $$\frac{d}{dt}\int_{V(t)} f(\underline{r},t)\,dV = \int_{V(t)} \frac{\partial f}{\partial t}\,dV + \int_{S(t)} (\underline{n}\cdot\underline{v}_s) f\,dS$$

surface velocity

- When can I "pull in" $\frac{d}{dt}$? when surface is constant in t.
- Notice similarities & differences with G.D.T.