Lecture 4 - Differential Equations

I. Intro to ODEs

* ordinary differential equations are equations that contain differentials w.r.t. one variable:

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}\right) \qquad (1)$$

* The objective is to find the function $y(x)$ that satisfies Eq. 1 and the corresponding initial or boundary conditions (more on that later).

* The value of n on the highest derivative is the order of the differential equation.

* We will focus on linear ODEs. They are of the form:

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\,y = h(x)$$

not a function of y

* If $h(x) = 0$, the ODE is said to be homogeneous. If not, the ODE is non-homogeneous.

* Classes of ODEs: we are going to look at the following classes of linear ODEs.

- First-order
  - separable
  - homogeneous
  - All other.
- N^{th} order
  - constant coefficient, homogeneous
  - constant coefficient, non-homogeneous
  - selected non-constant coefficient

II. First Order ODEs

A. Separable

* Separable 1^{st} order ODEs are of the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

* They can be solved by integrating

$$\int g(y)\, dy = \int f(x)\, dx$$

* These do not need to be linear

B. Homogeneous, linear 1^{st} order

* A linear, homogeneous 1^{st} order ODE has the form

$$\frac{dy}{dx} + a_1(x)\, y = 0$$

not constant coefficient

homogeneous

* This is separable, so we can solve it:

$$\frac{dy}{dx} = -y\,a_1(x)$$

$$\int \frac{1}{y}\,dy = \int -a_1(x)\,dx$$

$$\ln y = -\int a_1(x)\,dx + c$$

$$\boxed{y(x) = C\exp\left(-\int a_1(x)\,dx\right)}$$

different arbitrary constant

C. General linear 1st order

* In general, the linear 1st order ODE is

$$\frac{dy}{dx} + a_1(x)\,y = h(x)$$

* How find $y(x)$ now? D'Alembert came up with a guess. It turns out this strategy is useful again & again. (Basis for variation of parameters, for example):

$$y(x) = c(x)\,y_h(x)$$

homogeneous

"constant" becomes a function of x

$$= c(x)\exp\left(-\int a_1(x)\,dx\right)$$

* substitute into the above:

$$\underbrace{\frac{d}{dx}\left[c(x)\exp\left(-\int a_1\,dx\right)\right]} + a_1\,c(x)\exp\left(-\int a_1\,dx\right) = h(x)$$

$$c\,\underbrace{\frac{d}{dx}\left[\exp\left(-\int a_1\,dx\right)\right]} + \frac{dc}{dx}\exp\left(-\int a_1\,dx\right)$$

$$-\exp\left(-\int a_1\,dx\right)\underbrace{\frac{d}{dx}\int a_1\,dx}_{a_1(x)}$$

$$- c(x)\, a_1(x) \cancel{\exp\left(-\int a_1\, dx\right)} + \frac{dc}{dx} \exp\left(-\int a_1\, dx\right)$$

$$+ a_1(x)\, c(x) \cancel{\exp\left(-\int a_1\, dx\right)} = h(x)$$

$$\frac{dc}{dx} = \underbrace{\exp\left(\int a_1\, dx\right)}_{p(x)} h(x)$$

$$c(x) = \int p(x)\, h(x) + D$$

constant

$$y(x) = \left[\int p(x)\, h(x) + D\right] \exp\left(-\int a_1(x)\, dx\right)$$

$1/p(x)$

$$\boxed{y(x) = \frac{D}{p(x)} + \frac{1}{p(x)} \int p(x)\, h(x) \qquad p(x) = \exp\left(\int a_1(x)\, dx\right)}$$

III N^{th} order Linear ODEs

* I will focus on 2^{nd} order only. The procedure for higher order is similar, but more tedious

A. Homogeneous, Constant Coefficient 2^{nd} order ODE

* These ODEs have the form

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0 \qquad (\text{III.1})$$

not functions of x

homogeneous

* To solve these, we assume a solution of the form

$$y = e^{rx}$$

* Substituting into III.1:

$$\frac{dy}{dx} = re^{rx} \qquad \frac{d^2y}{dx^2} = r^2 e^{rx}$$

$$r^2 e^{rx} + a_1 r e^{rx} + a_2 e^{rx} = 0$$

common factor: e^{rx}

$$\boxed{r^2 + a_1 r + a_2 = 0}$$ ← "characteristic equation"

(will get r^3 for 3rd order, etc.)

* Quadratic equation gives solution:

$$r = -\frac{a_1}{2} \pm \frac{1}{2}\sqrt{a_1^2 - 4a_2}$$

* <u>case 1: $a_1^2 > 4a_2$</u> (2 real roots)

$$\boxed{y = c_1 e^{r_1 x} + c_2 e^{r_2 x}}$$ ← both roots satisfy ODE, so the sum does to.

"General solution"

"Fundamental set" of solutions

- It is sometimes convenient to express this solution using hyperbolic functions:

$$y = d_1 \sinh(r_1 x) + d_2 \cosh(r_2 x)$$

$$\sinh(rx) = \frac{e^{rx} - e^{-rx}}{2}, \qquad \cosh(rx) = \frac{e^{rx} + e^{-rx}}{2}$$

* case 2: $a_1^2 < 4a_2$

- get a negative sign under the radical
  ↳ 2 complex roots.

$$r = \lambda \pm i\mu \qquad \lambda = \frac{-a_1}{2}, \quad \mu = \sqrt{|a_1^2 - 4a_2|}$$

$$y = c_1 e^{(\lambda + i\mu)x} + c_2 e^{(\lambda - i\mu)x}$$

↖ usually a pain to leave complex, instead we use Euler's formula to write as sin / cos:

$$\sin(ax) = \frac{e^{iax} - e^{-iax}}{2i}$$

$$\cos(ax) = \frac{e^{iax} + e^{-iax}}{2}$$

$$\boxed{y = d_1 e^{\lambda x} \sin(\mu x) + d_2 e^{\lambda x} \cos(\mu x)}$$

* case 3: $a_1^2 = 4a_2$

- get 0 in the square root: only one solution

$$y = c_1 e^{\lambda x}$$

- Let's use our trick! Plug into (III.1)

$$y = c(x)\, e^{\lambda x}$$

product rule

$$\frac{dy}{dx} = c\lambda e^{\lambda x} + \frac{dc}{dx} e^{\lambda x}$$

$$\frac{d^2y}{dx^2} = c\lambda^2 e^{\lambda x} + 2\lambda \frac{dc}{dx} e^{\lambda x} + \frac{d^2c}{dx^2} e^{\lambda x}$$

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0$$

$$c\lambda^2 e^{\lambda x} + 2\lambda\frac{dc}{dx}e^{\lambda x} + \frac{d^2c}{dx^2}e^{\lambda x} + a_1 c\lambda e^{\lambda x} + a_1\frac{dc}{dx}e^{\lambda x}$$

$$+ a_2 c e^{\lambda x} = 0$$

- common factor $e^{\lambda x}$
- recall $\lambda = -a_1/2$ or $2\lambda = -a_1$

$$c\frac{a_1^2}{4} - \cancel{a_1\frac{dc}{dx}} + \frac{d^2c}{dx^2} - \frac{a_1^2}{2}c + \cancel{a_1\frac{dc}{dx}} + a_2 c = 0$$

$$\frac{d^2c}{dx^2} + \underbrace{\left(a_2 - \frac{a_1^2}{4}\right)}_{0,\ b/c\ \text{case 3}\ a_1^2 = 4a_2} c = 0$$

$$\frac{d^2c}{dx^2} = 0 \quad \text{or} \quad c(x) = d_1 x + d_2$$

$$\boxed{y(x) = d_1 x e^{\lambda x} + d_2 e^{\lambda x}}$$

* Finally take a look at table B-1 on p. 641 for a summary of these cases at nth order

B. Non-Homogeneous, Constant Coefficient, 2nd order ODEs

* These ODEs have the form:

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = h(x) \qquad (\text{III}.2)$$

* A fundamental result of linear ODEs says that:

$$y(x) = y_h(x) + y_p(x)$$

$y_h(x)$: homogeneous solution

$y_p(x)$: "particular" solution

* There are two ways to find the particular solution:

(1) Variation of parameters: This is our "trick" we've been using. We assume:

$$y(x) = c_1(x)\, y_1(x) + c_2(x)\, y_2(x)$$

The substitute & solve for c_1 & c_2. This always works, but is tedious.

(2) Method of undetermined coefficients

In this method we assume a form for y_p: $y_p(x) = \underset{\text{constant}}{c}\, \underset{\text{guess}}{g(x)}$.

we plug this in and solve for the unknown constant.

Table B.2 on p. 642 has guesses. Basically, people did variation of parameters to figure out good guesses.

* Example:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 3e^{2x}$$

Guess $y_p = ce^{2x}$

$$\frac{dy_p}{dx} = 2ce^{2x} \qquad \frac{d^2y_p}{dx^2} = 4ce^{2x}$$

$$4c\cancel{e^{2x}} - 3\cdot 2c\cancel{e^{2x}} - 4c\cancel{e^{2x}} = 3e^{2x}$$

$$(4-6-4)c = 3 \Rightarrow -6c = 3 \Rightarrow c = -1/2$$

$$\boxed{y_p = -\frac{1}{2}e^{2x}}$$

IV. N^{th} order, Non-constant coefficient ODEs

* There is no general procedure for greater than first order non-constant coefficient ODEs (Either homogeneous or not). Instead, there are some special cases that are relevant for transport.

A. Bessel's Equations

(1) $x\frac{d}{dx}\left(x\frac{dy}{dx}\right) + (m^2x^2 - v^2)y = 0$ Bessel's Eq.

(2) $x\frac{d}{dx}\left(x\frac{dy}{dx}\right) - (m^2x^2 + v^2)y = 0$ Modified Bessel's Eq.

v: a constant (non-negative integer)
m: a parameter

solutions to 1:

$$y = c_1 J_v(mx) + Y_v(mx)$$

Bessel Functions of 1st & 2nd kind of order v.

solutions to 2:

$$y = c_1 I_v(mx) + c_2 K_v(mx)$$

Modified Bessel Fn. of 1st & 2nd kind of order v.

* J_ν & Y_ν are kind of like sin & cos

* I_ν & K_ν are kind of like exp(+x), exp(-x)

* They may seem "weird", but you are just unfamiliar with them. sin/cos/exp are strange transcendental functions too.

* Like sin, cos, exp. there are useful relations between derivatives of these functions.

* Bessel's Equations arise in cylindrical coordinates. (sin/cos/exp → cartesian)

* Show plots.

B. Spherical Bessel Functions

* Just like Bessel's Eq & Mod. Bessels Eq that come from cylindrical coordinates, There are spherical Bessel's Equations & modified spherical Bessel's Equations

(3) $$\frac{d}{dx}\left(x^2 \frac{dy}{dx}\right) + \left[m^2 x^2 - n(n+1)\right] y = 0$$ spherical Bessel's Eq.

(4) $$\frac{d}{dx}\left(x^2 \frac{dy}{dx}\right) - \left[m^2 x^2 + n(n+1)\right] y = 0$$ Modified spherical B.E.

* usually, we only need order 0. ($n=0$).

In that case the solutions are:

(3) $$y = c_1 \frac{\cos(mx)}{mx} + c_2 \frac{\sin(mx)}{mx}$$

(4) $$y = c_1 \frac{\cosh(mx)}{mx} + c_2 \frac{\sinh(mx)}{mx}$$

* you can see expressions for the solution to the 1st order equations in terms of sin/cos/sinh/cosh in your book.

C. Other Functions

* There are other important non-constant coefficient ODEs in your book that lead to special functions. In particular, we will see:

  - Equidimensional equations
  - Error functions
  - Legendre Polynomials.

* we don't have time to go over them, so take a look.

V. Initial & Boundary Conditions

* In all of the above, we have left the solution as a "general solution" with unspecified constants.

* To find a physical solution we will need to solve for these constants using initial or boundary conditions

* Initial conditions are values of the function, or derivatives, that are specified at a single point.

  - Example: $\frac{d^2y}{dt^2} = -g$ $y(0) = 5$ ← initial height
    $y'(0) = 0$ ← initial speed.

    ↑ kinematics. Ball falling.

  - Initial conditions are typically associated with time derivatives. Usually specified at $t=0$.

* Boundary conditions are values of the function, or derivatives, that are specified at more than one point.

  - Example: $\frac{d^2y}{dx^2} = 0$ $y(1) = 5$ ← values at domain boundaries
    $y(0) = 0$

    ↑ diffusion equation, steady state

  - Boundary conditions are typically associated with spatial derivatives, and are usually specified at domain edges.

* Problems with ICs are called "Inital Value Problems" (IVPs). Problems with BCs are called "Boundary value problems (BVPs).

* IVPs that are linear have an existence & uniqueness theorem. This says that there is one and only one solution to a linear IVP. (superposition principle applies still to general solution).

* For BVPs however, it is a bit more subtle.

BVPs can have zero, one, or infinitely many solutions

* I don't want to prove this, but there is a useful analogy to linear algebraic systems.

* A homogeneous BVP is like a homogeneous L A problem

$\mathcal{L} y = 0$ $\qquad\qquad$ $\underline{A} \cdot \underline{y} = \underline{0}$

differential operator

- "function - vector" analogy.
- we will see this later

* In Linear Algebra if all of the rows of A linearly independent, then $y=0$ is the only solution. If not, then there are infinite solutions.

* In a homogeneous BVP, there is something similar. The solution can either be the "trivial" solution $y=0$, or there are infinitely many.

* Example:

Note: a homogeneous BVP must have BCs that are zero to be homogeneous.

$$\frac{d^2y}{dx^2} + 2y = 0 \qquad y(0)=0, \quad y(\pi)=0$$

$$y = c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x)$$

$$y(0) = c_1 = 0$$

not zero.

$$y(\pi) = c_2 \sin(\sqrt{2}\pi) = 0$$

$$c_2 = 0$$

trivial solution only.

$$\boxed{y(x) = 0}$$

* Example 2:

$$\frac{d^2y}{dx^2} + y = 0 \qquad y(0) = 0, \quad y(\pi) = 0$$

$$y = c_1 \cos(x) + c_2 \sin(x)$$

$$y(0) = c_1 = 0$$

$$y(\pi) = c_2 \sin(\pi) = 0$$

sin π is always 0! c_2 can be anything!

$$\boxed{y(x) = c_2 \sin x}$$