Lecture 7 – Conservation Equations

* Remember a few days ago, I said transport equations require two pieces:

    - Constitutive Laws
    - Balance Equations

* We have done constitutive laws in chapter 1. These laws are empirical and material specific. They give us relationships for fluxes

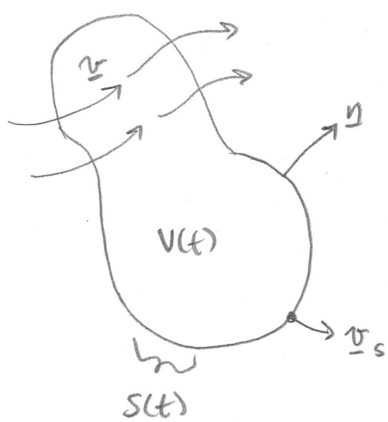
* We will now focus on balance equations. They are general & universal for conserved quantities.

* We will then combine the two together to get transport equations for:

    - total mass
    - species mass
    - energy

I. General Balances

* Consider the following arbitrary control volume (C.V.)



V(t): volume
S(t): surface area
$\underline{n}$: outward unit normal
$\underline{v}$: fluid velocity
$\underline{v}_s$: surface velocity (of C.V.)

* A C.V. is a mathematically defined region of study like the "system" in thermo. It isn't magic, but a "smart" choice can simplify the problem.

* We want to do a balance of some property, B, in the control volume. This is basically just accounting, but it's fancy because it can change in space and time.

(*) Note: This is a "general" balance because it isn't specific to a single property. This abstraction can make it hard to understand. If it helps, imagine B = mass.

* Accounting Balance:

- accumulation = in − out + generation
- let $b(\underline{r}, t)$ be the quantity B per volume at point $\underline{r}$ and time t in our C.V.

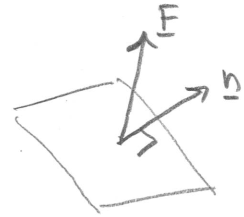
accumulation: $\frac{d}{dt} \int_{V(t)} b(\underline{r}, t)\, dV$

- let $B_V(r, t)$ be the rate of creation of B per unit volume

generation: $\int_{V(t)} B_V(r, t)\, dV$

- let $\underline{F}$ be the total flux(*) of B accross the control surface.

(*) More about fluxes in a bit!



net rate of B in? $-\underline{n} \cdot \underline{F}$

↑ outward unit normal

<u>in-out</u> : $-\int_{S(t)} \underline{n} \cdot \underline{F} \, dS$

- But wait, the surface can move. If it gets bigger (smaller) then we will add (subtract) $b(r,t)$ from the C.V.

  - volumetric flow rate: $v_s \cdot$ Area
  - in differential form: $\underline{n} \cdot \underline{v}_s \, dS$

<u>in-out (2)</u>: $+\int_{S(t)} b(\underline{r},t) \, \underline{n} \cdot \underline{v}_s \, dS$

- Put it all together:

$$\frac{d}{dt} \int_{V(t)} b(r,t) \, dV = -\int_{S(t)} \underline{n} \cdot \underline{F} \, dS + \int_{S(t)} b \underline{n} \cdot \underline{v}_s \, dS + \int_{V(t)} B_V \, dV$$

* we can simplify this with Leibniz' rule (A.5-9, p. 623):

$$\frac{d}{dt} \int_{V(t)} b \, dV = \int_{V(t)} \frac{\partial b}{\partial t} \, dV + \int_{S(t)} b \underline{n} \cdot \underline{v}_s \, dS$$

(how differentiate an integral)

- substitute into the LHS of the above & the $\underline{n} \cdot \underline{v}_s$ terms cancel

$$\boxed{\int_{V(t)} \frac{\partial b}{\partial t} \, dV = -\int_{S(t)} \underline{n} \cdot \underline{F} \, dS + \int_{V(t)} B_V \, dV} \quad (1)$$

* Comments:

- General integral balance: sophisticated accounting
- Useful for "macroscopic analysis" (e.g. mechanical energy balances in fluids)

* We can use the general integral balance to get a general differential balance!

* Assume that $V(t) = V$ & $S(t) = S$. (This isn't strictly necessary, but it simplifies the derivation.)

* Recall Gauss' Divergence Theorem (A.5-2, p. 621)

$$\int_S \underline{n} \cdot \underline{F}\, dS = \int_V \underline{\nabla} \cdot \underline{F}\, dV$$

- substitute into gen. integral balance

$$\int_V \frac{\partial b}{\partial t}\, dV = -\int_V \underline{\nabla} \cdot \underline{F}\, dV + \int_V B_V\, dV$$

$$\Rightarrow \int_V \left[\frac{\partial b}{\partial t} + \underline{\nabla} \cdot \underline{F} - B_V \right] dV = 0$$

volume is arbitrary, so the integrand must be zero.

$$\boxed{\frac{\partial b}{\partial t} = -\underline{\nabla} \cdot \underline{F} + B_V} \qquad (2)$$

* comments:

- General differential balance: balance at every point in space
- PDE to solve for $b(\underline{r}, t)$
- Very general. Good for any scalar quantity, b, as long as the domain is continuous.

II. Example: conservation of total mass

* For total mass:

- $B = m \longrightarrow b = \frac{m}{V} = \rho$ ← density
- $\underline{F} = \rho \underline{v} \longrightarrow$ mass flux is equal to $\frac{\text{"mass flow rate"}}{\text{volume}}$
- $B_V = 0 \longrightarrow$ no mass created or destroyed.

$$\frac{\partial b}{\partial t} = -\nabla \cdot F + B_V$$

$$\frac{\partial \rho}{\partial t} = -\underline{\nabla} \cdot (\rho \underline{v}) + 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = 0}$$

"continuity equation"

- Mass conservation at a point in space.

* A common simplification is $\rho = \text{const}$.

This simplifies continuity to:

$$\boxed{\underline{\nabla} \cdot \underline{v} = 0}$$

"incompressible" continuity Eq.

(Extra space b/c added this page)

III. Alternate Forms of the balances

* The general balances have different forms they can be written in.
* One way to write them involves dividing the total flux, $\underline{F}$, into a convective and diffusive part:

$$\underbrace{\underline{F}}_{\text{total flux}} = \underbrace{b\underline{v}}_{\text{convective flux}} + \underbrace{f}_{\text{diffusive flux}}$$

example: $\underbrace{\underline{n}_i}_{\text{total flux of } i} = \underbrace{\rho_i \underline{v}}_{\text{"mass flow rate"}} + \underbrace{\underline{j}_i}_{\text{diffusive flux from Fick's law}}$ $\quad b = \rho_i$ ← species mass

(*) For total mass, $\underline{f} = 0$. No diffusive flux.

* The diffusive flux is due to molecular motion causing transport.
* The convective flux is due to bulk fluid flow carrying the property with it.
* Subbing into our differential balance gives

$$\underline{\nabla} \cdot F = \underline{\nabla} \cdot (b\underline{v}) + \underline{\nabla} \cdot \underline{f}$$

$$\boxed{\frac{\partial b}{\partial t} + \underline{\nabla} \cdot (b\underline{v}) = -\underline{\nabla} \cdot f + B_V} \quad (3)$$

* Another way to write balances uses the material derivative:

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \underline{v} \cdot \underline{\nabla}$$

- Rate of change seen by an observer moving with the fluid velocity (a material element)
- Same as a total derivative, with spatial derivatives set to fluid velocity:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial}{\partial z}$$

(only write if they need to see it)

* We can write the general property balance using the material derivative as well:

Eq 3: $$\frac{\partial b}{\partial t} + \underline{\nabla} \cdot (b\underline{v}) = -\underline{\nabla} \cdot \underline{f} + B_V$$

Let $\hat{B} = b/\rho \Rightarrow b = \rho \hat{B}$ ← "specific" B

$$\frac{\partial}{\partial t}(\rho \hat{B}) + \underline{\nabla} \cdot (\rho \hat{B} \underline{v}) = -\underline{\nabla} \cdot \underline{f} + B_V$$

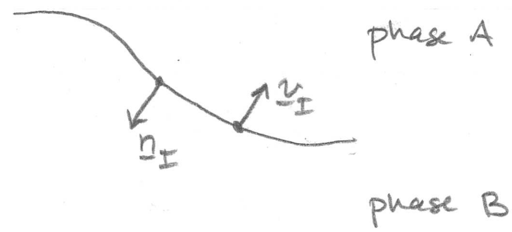
$$\hat{B}\underbrace{\left[\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v})\right]}_{\substack{0 \\ \text{(continuity)}}} + \rho \underbrace{\left[\frac{\partial \hat{B}}{\partial t} + \underline{v} \cdot \underline{\nabla} \hat{B}\right]}_{\frac{D\hat{B}}{Dt}} = -\underline{\nabla} \cdot \underline{f} + B_V$$

$$\boxed{\rho \frac{\partial \hat{B}}{\partial t} = -\underline{\nabla} \cdot f + B_V}$$

IV. Conservation Equations at Interfaces

* What happens if $b(\underline{r}, t)$ is not continuous? This happens at boundaries. Balances put a constraint on boundary conditions.

* Special control volume that includes a boundary in it:



$\underline{n}_I$: unit normal of interface (points to B by convention)

$\underline{v}_I$: velocity of interface

B_S : generation at surface

$\underline{F}$: total flux

* Skipping a very similar derivation (see §2.2, p.31) we get the equation:

$$\boxed{\left[(\underline{F} - b\underline{v}_I)_B - (\underline{F} - b\underline{v}_I)_A \right] \cdot \underline{n}_I = B_S}$$

↑ flux relative to the interface on the B side

↑ flux relative to the interface on the A side

↑ generation at the interface.

* Using convective and diffusive fluxes, this can be written as:

$$\boxed{\left[(\underline{f} + b(\underline{v} - \underline{v}_I))_B - (\underline{f} + b(\underline{v} - \underline{v}_I))_A \right] \cdot \underline{n}_I = B_S}$$

$\underline{v}$: fluid velocity , recall $\underline{F} = b\underline{v} + \underline{f}$