Lecture 10 - Solving 1D Transport Equations

I. Motivation and Overview

A. Motivation

* Transport equations are (in general) non-linear PDEs. They are hard to solve. We'd like to simplify them as much as possible.

* Why not just solve the whole non-linear PDE on a computer?

  - We can and do sometimes. That isn't easy either. Before ≈1990, not really feasible. Before ≈2005, not convenient (Matlab, comsol).
  - Analytical solutions provide insight. "ABC's" of transport.
  - Analytical solutions can be used to check numerical solutions on computer
  - Analytical solutions are faster to compute.
  - Analytical solutions teach us more. More to learn from.

Analytical solution	⟵⟶	Numerical solution
Faster		Slower
More Insight		More accurate, Quantitative
More Qualitative		More can be solved
More approximations		Fewer approximations
Limited methods		Compare to experiment

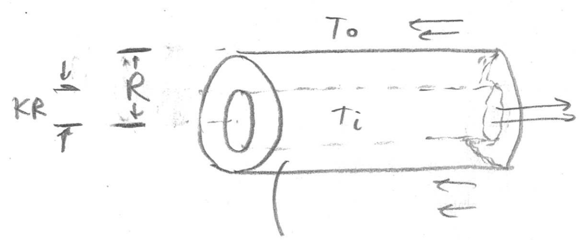
* In real life, we do both! If an analytical solution exists, it helps a lot! Both have their place.

B. Steps to solutions

1. Draw a picture. ID geometry.
   ID balance equation & boundary conditions
2. Write balance equation in right coordinate system.
   Simplify equation until it is one-dimensional.
3. Solve the math problem
4. Check physical interpretation. what does it mean?
   Does it make sense?

II. Examples

A. Heat conduction in a tube wall



k: conductivity of tube

$\hat{C}_p$: heat capacity

- Tube. $OD = R$
  $ID = KR$
- Fluid flowing very fast. $T_{inner} = T_i$
  $T_{outer} = T_o$ } Boundary conditions
  $T(r=R, \theta, z) = T_o$
  $T(r=kR, \theta, z) = T_i$

* Find temperature distribution in wall. Could use find heat flux (but we don't have time).

* Balance Equation: Energy equation in cylindrical coordinates

$$\rho \hat{C}_p \frac{DT}{Dt} = k \nabla^2 T + H_v \qquad \text{(p. 37, table 2-3)}$$

$$\rho \hat{C}_p \left[\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right] = k \left[\underline{\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right)} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right] + H_v$$

- steady : $T \neq T(t)$, $\frac{\partial T}{\partial t} = 0$
- one-dimensional : $T = T(r)$ $\quad \frac{\partial T}{\partial \theta} = \frac{\partial T}{\partial z} = 0$
- Solid wall : $v_r = v_\theta = v_z = 0$
- No heat generation : $H_v = 0$

$\Rightarrow$ only underlined terms survive

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) = 0$$

* Solve math problem:

$$\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) = 0 \qquad T(R) = T_o \quad T(kR) = T_i$$

- Separable, linear, 2nd order homogeneous ODE with Dirichlet BC's.

$$r \frac{\partial T}{\partial r} = c_1 \quad \Rightarrow \quad \frac{\partial T}{\partial r} = \frac{c_1}{r} \quad \Rightarrow \quad T = c_1 \ln r + c_2$$

- Solve for BC's :

$$T(R) = c_1 \ln R + c_2 = T_o \qquad (1)$$

$$T(kR) = c_1 \ln kR + c_2 = T_i \qquad (2)$$

$$(1) - (2) = c_1 \ln (R/kR) = T_o - T_i$$

$$c_1 = \frac{T_o - T_i}{-\ln(k)} = \frac{T_i - T_o}{\ln k}$$

$$(1) : \quad (T_i - T_o)\frac{\ln R}{\ln k} + c_2 = T_o$$

$$c_2 = T_o - \frac{\ln R}{\ln k}(T_i - T_o)$$

$$T(r) = \frac{T_i - T_o}{\ln k} \ln r - \frac{\ln R}{\ln k}(T_i - T_o) + T_o$$

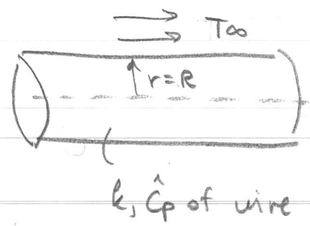
$$\boxed{T(r) = T_o + \frac{(T_i - T_o)}{\ln k} \ln\left(\frac{r}{R}\right)}$$

* get q_r by

$$q_r = -k \left.\frac{\partial T}{\partial r}\right|_{r=R}$$

* Does this make sense? Look at python plot.

B. Electrically Heated wire (Ex. 3.2-5)



- wire diameter: R
- Fluid flowing w/ heat transfer coeff, h.
- Heating rate due to electrical heating: H_v.

- Boundary conditions?

  - Symmetry at $r=0$

$$\left.\frac{dT}{dr}\right|_{r=0} = 0$$

  - Newton's law of cooling / convection at $r=R$

$$-k \left.\frac{dT}{dr}\right|_{r=R} = h\left[T(R) - T_\infty\right]$$

* Balance equation: Energy equation in cylindrical coords. (same as before)

$$\rho\hat{C}_p \frac{DT}{Dt} = k\nabla^2 T + H_v$$

$$\rho\hat{C}_p\left[\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + v_z\frac{\partial T}{\partial z}\right] = k\left[\underline{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right] + \underline{H_v}$$

only surviving terms

- steady, $\left(\frac{\partial T}{\partial t}=0\right)$; one-dimensional $(T=T_r)$, $(T \neq T(z) \neq T(\theta))$; solid wire $(\underline{v}=0)$

$$\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + H_v = 0$$

* Solve Math problem

$$\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + H_v = 0 \qquad \left.\frac{\partial T}{\partial r}\right|_{r=0} = 0$$

$$\left.\frac{\partial T}{\partial r}\right|_{r=R} = -\frac{h}{k}\left[T(R) - T_\infty\right]$$

- separable, linear, 2^{nd} order, non-homogeneous ODE with Robin condition. (Harder problem, a bit.)

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = -r\frac{H_v}{k}$$

$$r\frac{\partial T}{\partial r} = -\frac{r^2}{2}\frac{H_v}{k} + C_1 \Rightarrow \frac{\partial T}{\partial r} = -\frac{r}{2}\frac{H_v}{k} + \frac{C_1}{r}$$

$$T = -\frac{r^2}{4}\frac{H_v}{k} + C_1 \ln r + C_2$$

(*) Same general solution, but w/ particular solution due to H_v.

- Apply BC's to find C_1, C_2

BC 1: $\left.\frac{dT}{dr}\right|_{r=0} = 0 \qquad \lim_{r\to 0}\frac{dT}{dr} = \frac{C_1}{r} \to \infty, \quad C_1 = 0$

$$\left.\frac{dT}{dr}\right|_{r=R} = -\frac{H_v R}{2k}$$

$$T(r=R) = \frac{-R^2 \cdot H_v}{4k} + C_2$$

plug into BC 2

BC 2: $\left.\frac{\partial T}{\partial r}\right|_{r=R} = -\frac{h}{k}\left[T(R) - T_\infty\right]$

Solve for C_2

$$\frac{-H_v \cdot R}{2k} = -\frac{h}{k}\left[-\frac{H_v R^2}{4k} + C_2 - T_\infty\right]$$

$$+\frac{H_v R}{2h} = -\frac{H_v R^2}{4k} + C_2 - T_\infty$$

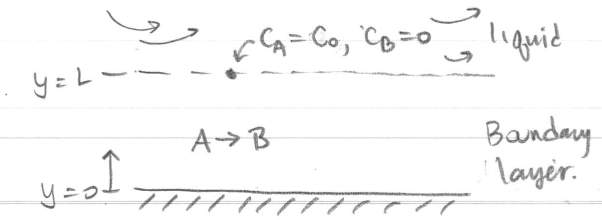
$$C_2 = T_\infty + \frac{H_v R^2}{4k} + \frac{H_v R}{2h}$$

$$T = -\frac{H_v r^2}{4k} + \frac{H_v R^2}{4k} + \frac{H_v R}{2h} + T_\infty$$

$$\boxed{T = T_\infty + \underbrace{\frac{H_v R}{2h}}_{\text{Temp. at surface}} + \underbrace{\frac{H_v R^2}{4k}\left(1 - \frac{r^2}{R^2}\right)}_{\text{Temp distribution from center to wire edge.}}}$$

* Does it make sense? See plot!

C. Irreversible Homogeneous Reaction in a liquid (Ex. 3.2-1)



- Consider close to the boundary in a liquid. (boundary layer)
  - L set by external fluid flow problem, given
  - $\underline{v} \approx 0$ in B.L.

- Homogeneous reaction : $A \to B$

$$R_{V,A} = -kC_A$$
$$R_{V,B} = kC_A$$

- Pseudo binary solution

$$C_A, C_B \ll C_S$$

- Boundary conditions :

$$C_A(y=L) = C_0 \quad , \quad C_B(y=L) = 0$$

$$\underline{J}_A \cdot \underline{n}\big|_{y=0} = 0 \quad , \quad \underline{J}_B \cdot \underline{n}\big|_{y=0} = 0$$

$$\frac{dc_A}{dy}\bigg|_{y=0} = 0 \qquad \frac{dc_B}{dy}\bigg|_{y=0} = 0$$

* Balance Equations : Pseudo binary species mass balance in Cartesian coordinates

$$\frac{Dc_A}{Dt} = D_A \nabla^2 C_A + R_{V,A}$$

← Let's just solve C_A
- time
- need C_A to find C_B b/c of reaction.

$$\frac{Dc_B}{Dt} = D_B \nabla^2 C_B + R_{V,B}$$

(Table 2-4)

$$\frac{\partial c_A}{\partial t} + v_x \frac{\partial c_A}{\partial x} + v_y \frac{\partial c_A}{\partial y} + v_z \frac{\partial c_A}{\partial z} = D_A \left[\frac{\partial^2 c_A}{\partial x^2} + \underline{\frac{\partial^2 c_A}{\partial y^2}} + \frac{\partial^2 c_A}{\partial z^2} \right] \underline{-kc_A}$$

- steady : $\dfrac{\partial c_A}{\partial t} = 0$

- stagnant : $v_x = v_y = v_z = 0$

- one-dimensional :

$$c_A \neq c_A(x) \neq c_A(z)$$

⇒ only underlined terms survive

* Careful with velocities in mass transfer problems. See Supp. notes. Diffusive flux can cause bulk flow:

$$\sum_i \underline{N}_i = c\underline{v}$$

* Solve math problem:

$$D_A \frac{\partial^2 c_A}{\partial y^2} - k c_A = 0 \qquad c_A(y=L) = C_o$$
$$\frac{dc_A}{dy}(y=0) = 0$$

- 2^{nd} order, linear, constant coefficient ODE w/ 1 Dirichlet / 1 Neumann condition.

$$\frac{\partial^2 c_A}{\partial y^2} - \frac{k}{D_A} c_A = 0$$

Let $C_A = Ce^{ry}$ $\rightarrow$ $r^2 - \frac{k}{D_A} = 0$ $\rightarrow$ $r = \pm\sqrt{\frac{k}{D_A}}$

Let $\lambda = \sqrt{\frac{D_A}{k}}$ $\rightarrow$ $r = \pm 1/\lambda$ (λ is a length, b/c k is 1/time)

$$C_A = c_1 \sinh(y/\lambda) + c_2 \cosh(y/\lambda)$$

- Solve for constants

$$c_A(L) = c_1 \sinh(L/\lambda) + c_2 \cosh(L/\lambda) = c_o$$

$$\frac{dc_A}{dy}(y=0) = \frac{c_1}{\lambda}\underbrace{\cosh(0)}_{1} + \frac{c_2}{\lambda}\underbrace{\sinh(0)}_{0} = 0$$

$c_1 = 0$, so

$c_2 = c_o / \cosh(L/\lambda)$

$$\boxed{c_A(y) = c_o \frac{\cosh(y/\lambda)}{\cosh(L/\lambda)}, \quad \lambda = \sqrt{\frac{D_A}{k}}}$$

* Does this make sense? See python plot.