Lecture 11 - Scaling

I. Intro & Motivation

* Solve ODEs (& PDEs) is hard. We want ways to:
  - Avoid doing this
  - learn as much as we can from our answers
  - Simplify ODEs / PDEs to easier ones to solve using what we know about the physical characteristics of the system.

* We can accomplish all of the above using "Scaling techniques." Scaling has two parts:
  - Order of magnitude analysis (OOM)
  - Non-dimensionalization.

* In OOM, we estimate the size or scale of terms in our differential equation. This allows us to make educated guesses about answers, without needing to solve a differential equation.

* In non-dimensionalization, we re-write our equation with the minimum number of parameters possible. This allows us to summarize physical behavior in very concise and insightful ways.
  (ex. Reynolds number: low Re = laminar, high Re = turbulent).

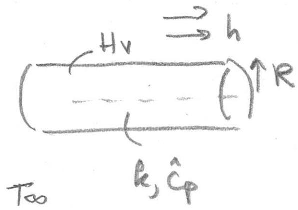
* Combined together, we can learn a lot from our equations before and after we solve them, and even use the info to make rational simplifications and approximations

II. Order-of-magnitude Analysis

* In an order of magnitude (OOM) analysis, we are trying to use the problem to estimate a scale for all of the terms in the equation.
  - How big (small) are they?
  - which ones matter?

* This is most easily demonstrated with an example:

Example: OOM Analysis of heated wire



$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = -\frac{H_v}{k}$$

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0 \qquad \left.\frac{\partial T}{\partial r}\right|_{r=R} = -\frac{h}{k}\left[T(R) - T_\infty\right]$$

T_s: surface temp.

$R = 10^{-3}\,m$ $\quad T_\infty = 350K$

$k = 10^{2}\; W/mK$ $\quad H_v = 10^7\; W/m^3$

$\hat{C}_p = 10^3\; J/kgK$ $\quad h = 10^4\; W/m^2K$

$T_c = T(0)$

- what is the value of the surface temperature?
- what is " " " " centerline temperature?
- Can we do this w/out solving the ODE?
- Step 1: use finite differences on the derivatives.

$$\frac{\partial T}{\partial r} = \frac{\Delta T}{\Delta r} \qquad (\text{replace } \partial \text{ with } \Delta)$$

$$\frac{1}{r}\frac{1}{\Delta r}\left(r\frac{\Delta T}{\Delta r}\right) = -\frac{H_v}{k} \qquad \left.\frac{\Delta T}{\Delta r}\right|_{r=0} = 0$$

$$\left.\frac{\Delta T}{\Delta r}\right|_{r=R} = -\frac{h}{k}\left[T_s - T_\infty\right]$$

- step 2: plug in values from the problem
  - The symbol '$\sim$' means "is the order of magnitude of".

$r \sim R$ (so $r \approx 0.3R$ to $3R$)

$\Delta r \sim R$

$$\frac{\Delta T}{\Delta r} \sim \frac{T(R)-T(0)}{R} = \frac{T_s - T_c}{R}$$

- plug into diff. eq. & BC.

$$\frac{1}{R^2}\left(R\,\frac{T_s-T_c}{R}\right) = \frac{-H_v}{k} \qquad \text{(ODE)}$$

$$\frac{T_s-T_c}{R} = -\frac{h}{k}(T_s - T_\infty) \qquad \text{(BC 2)}$$

- solve :

$$\frac{T_s-T_c}{R^2} = \frac{-H_v}{k} \Rightarrow T_c = T_s + \frac{H_v R^2}{k}$$

$$\frac{T_s-T_c}{R} = -\frac{h}{k}(T_s-T_\infty) \Rightarrow \frac{-H_v R}{k} = -\frac{h}{k}(T_s - T_\infty)$$

$$\Rightarrow T_s = T_\infty + \frac{H_v R}{h}$$

$$\boxed{T_s = T_\infty + \frac{H_v R}{h}} \longrightarrow \frac{H_v R}{h} = 1K \text{ (use params)}$$

$$\boxed{T_c = T_\infty + \frac{H_v R^2}{k} + \frac{H_v R}{h}} \longrightarrow \frac{H_v R^2}{k} = 0.1\,K \text{ (use params)}$$

- Full solution? (see lec. 10, p.5)

$$T(R) = T_\infty + \frac{H_v R}{2h}$$

$$T(0) = T_\infty + \frac{H_v R}{2h} + \frac{H_v R^2}{4k}$$

(*) Same, but our new one is missing constants!

* Comments

- Our OOM analysis gives us an easy way to estimate quantities w/out solving diff. eqs.
- we can tell how big/small our numbers are. Maximum information with minimum mathematical effort/sophistication.

III. Dimensional Analysis

* Dimensional analysis tells us how many degrees of freedom we have in our equations. This lets us figure out the minimum number of parameters we need to describe the physical behavior of our system.

* A fundamental theorem of dimensional analysis tells us if we have n variables and m fundamental dimensions, then we have (n−m) degrees of freedom (dimensionless groups). (Buckingham Pi theorem)

* Example: Dimensional analysis of heated wire

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = -\frac{H_v}{k} \qquad \left.\frac{\partial T}{\partial r}\right|_{r=0} = 0 \; , \; \left.\frac{\partial T}{\partial r}\right|_{r=R} = \frac{-h}{k}\left[T(R) - T_\infty\right]$$

\# of variables : $r, T, H_v, k, h, T_\infty, R = 7$ (no $\rho, \hat{C}_p$)

\# dimensions : temperature, length, mass, time = 4
(Energy is a combo of len, mass, time)

\# degrees of freedom : $7-4 = 3$

* In practice, we find these degrees of freedom by making our equations dimensionless. We do this by rescaling the variables. This will leave us with (n−m) dimensionless groups.

* How do we pick the right scales? We use the quantities from our OOM analysis! Doing this (putting OOM + dimensional analysis) gives us a scaling analysis of our problem

* Example: Non-dimensionalize the heated wire problem

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{-H_v}{k}, \quad \left.\frac{\partial T}{\partial r}\right|_{r=0}=0, \quad \left.\frac{\partial T}{\partial r}\right|_{r=R} = \frac{-h}{k}\left[T(R)-T_\infty\right]$$

Step 1: redefine independent & dependent variable using scales from OOM analysis

scale: $\Delta T \sim T_s - T_c \sim \frac{-H_v R^2}{k}$

$r \sim R$

define: $\theta = \frac{T - T_\infty}{T_s - T_c} = \frac{-k}{H_v R^2}(T - T_\infty)$

$\eta = r/R$

Step 2: substitute θ & η into ODE. Arrange so the equation (& BCs) are dimensionless:

$$r = \eta R, \quad T = -\frac{H_v R^2}{k}\theta + T_\infty$$

ODE:

$$\frac{1}{\eta R}\frac{\partial}{\partial(\eta R)}\left[\eta \cancel{R}\frac{\partial\left(-\frac{H_v R^2}{k}\theta + \cancel{T_\infty}\right)}{\partial(\eta R)}\right] = -\frac{H_v}{k}$$

$$\frac{1}{\cancel{R^2}}\frac{1}{\eta}\frac{\partial}{\partial\eta}\left[\eta\cdot\left(\frac{-H_v \cancel{R^2}}{k}\right)\frac{\partial\theta}{\partial\eta}\right] = \frac{-H_v}{k}$$

$$-\frac{H_v}{k}\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\theta}{\partial\eta}\right) = \frac{-H_v}{k}$$

$$\boxed{\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\theta}{\partial\eta}\right) = 1}$$

BC1:
$$\left.\frac{\partial\left(\frac{-H_vR^2}{k}\theta+T_\infty\right)}{\partial(\eta R)}\right|_{\eta R=0}=0$$

$$\boxed{\left.\frac{\partial\theta}{\partial\eta}\right|_{\eta=0}=0}$$

BC2:
$$\left.\frac{\partial\left(\frac{-H_vR^2}{k}\theta+T_\infty\right)}{\partial(\eta R)}\right|_{\eta R=R}=\frac{-h}{k}\left[\frac{-H_vR^2}{k}\theta(\eta R=R)+\cancel{T_\infty}-\cancel{T_\infty}\right]$$

$$\left.\frac{-H_vR^2}{k}\cdot\frac{1}{R}\frac{\partial\theta}{\partial\eta}\right|_{\eta=1}=\frac{hH_vR^2}{k^2}\theta(\eta=1)$$

$$\left.\frac{\partial\theta}{\partial\eta}\right|_{\eta=1}=-\underbrace{\frac{hR}{k}}\theta(1)$$

Biot #: $Bi=\frac{hR}{k}$

$$\boxed{\left.\frac{\partial\theta}{\partial\eta}\right|_{\eta=1}=-Bi\,\theta(1)\qquad Bi=\frac{hR}{k}}$$

* Comments

- 3 dimensionless groups: θ, η, Bi
- Dimensionless groups tell us what physics matters!
  - $Bi \ll 1 \Rightarrow hR \ll k \Rightarrow$ convection << thermal conduction
  - $Bi \gg 1 \Rightarrow hR \gg k \Rightarrow$ convection >> thermal conduction
- Because of OOM analysis, $\theta \sim 1$ & $\eta \sim 1$. All of the physical differences are built into the other groups that come out of the scaling analysis. This always happens.
- Small ($\ll 1$) or large ($\gg 1$) dimensionless groups (e.g, Bi) allow us to make rational simplifications to our ODE. Maybe we get an easier one!

* See python plot for plots of dimensionless solution