Lecture 12 – Simplifying Transport Equations

I. Basis for simplifications

* Implicit in our discussion in lecture 10 (on solving 1D equations) was the fact that our equations did indeed simplify to 1D. Today we are going to revisit that idea.

* We are going to be a bit more careful, and we are going to rely on ideas from the last lecture on scaling to do this.

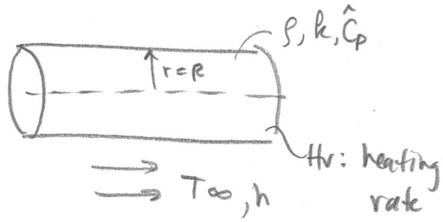
* Our book lists four different reasons for getting rid of terms in a transport equation:

1. Symmetry
2. Differing length scales
3. Differing resistance scales
4. Differing time scales.

} Note these three have to do with different scales. Need "scaling analysis"

* It is easier to talk about these four reasons in terms of an example, rather than abstractedly. Let's return to our heated-wire example.

* Example: Heated wire



$$\left.\frac{dT}{dr}\right|_{r=0} = 0$$

$$\left.\frac{dT}{dr}\right|_{r=R} = \frac{-h}{k}\left[T(R) - T_\infty\right]$$

$$\rho \hat{C}_p \frac{DT}{Dt} = k \nabla^2 T + H_v$$

$$\rho \hat{C}_p \left[\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right] = k \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right] + H_V$$

- First, recall that the wire is solid, so $v_r = v_\theta = v_z = 0$

1. Symmetry

- we know that the problem is cylindrically symmetric. That means $T(r,\theta,z) = T(r,\theta',z)$ or $\frac{\partial T}{\partial \theta} = 0$.
- Symmetry is the main reason for having & using curvilinear coordinates. (Remember, I said this when we learned about them in Lec 2.)
- Because our PDE is non-linear, there may be multiple solutions: some that are symmetric and some that aren't. For example: laminar & turbulent pipe flow.

2. Differing length scales

- If one length scale is much smaller, or larger, than the others, that can lead to simplifications.
- Consider an OOM analysis of the steady version of the energy equation above:

$$0 = \underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)}_{\sim \frac{\Delta T}{R^2}} + \underbrace{\frac{\partial^2 T}{\partial z^2}}_{\sim \frac{\Delta T}{L^2}}$$

T_c: center temp.
$\Delta T \sim T_c - T_\infty$

- let $\theta = \frac{T - T_\infty}{\Delta T}$, $\tilde{r} = r/R$, $\tilde{z} = z/L$

- non dimensionalize:

$$0 = \frac{1}{\tilde{r}R}\frac{\partial}{\partial(\tilde{r}R)}\left[\tilde{r}R\,\frac{\partial(\theta\cdot\Delta T)}{\partial(\tilde{r}R)}\right] + \frac{\partial^2(\theta\,\Delta T)}{\partial(\tilde{z}L)^2}$$

$$0 = \frac{\Delta T}{R^2}\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial \theta}{\partial \tilde{r}}\right) + \frac{\Delta T}{L^2}\frac{\partial^2 \theta}{\partial \tilde{z}^2}$$

$$0 = \frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial \theta}{\partial \tilde{r}}\right) + \frac{R^2}{L^2}\frac{\partial^2 \theta}{\partial \tilde{z}^2}$$

- when $R \ll L$, the temperature gradient along the z-direction is small. we can then neglect it!

- There are "edge effects" in the z-dir if we get to the end of the wire where $L \sim R$.

3. Differing resistance scales

- Remember, last time we non-dimensionalized our 1D problem

$$\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = -H_v \qquad \left.\frac{\partial T}{\partial r}\right|_{r=0} = 0, \qquad \left.\frac{\partial T}{\partial r}\right|_{r=R} = -\frac{h}{k}[T(R) - T_\infty]$$

$$\theta = \frac{T - T_\infty}{T_s - T_c}, \quad \eta = r/R$$

$$\frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial \theta}{\partial \eta}\right) = 1 \qquad \left.\frac{\partial \theta}{\partial \eta}\right|_{\eta=0} = 0 \qquad \left.\frac{\partial \theta}{\partial \eta}\right|_{\eta=1} = -Bi\,\theta(1)$$

$$Bi = hR/k$$

- The Biot # is a ratio of resistances.

resistance to heat transfer in the fluid : $1/hR$

resistance " " " " " solid : $1/k$

(a resistance in my definition here is $\frac{1}{\text{conductivity}}$)

$$Bi = \frac{\text{solid resistance}}{\text{fluid resistance}} = \frac{1/k}{1/hR} = hR/k$$

- When $Bi \ll 1$, the resistance in the solid is small.

Now:

$$\left.\frac{d\theta}{d\eta}\right|_{\eta=1} \approx 0$$

- This makes the solution to θ be a constant!

$$\theta = \frac{\eta^2}{4} + c_1 \ln \eta + c_2 \quad \text{(from lec. 10)}$$

$$\frac{d\theta}{d\eta} = \frac{\eta}{2} + \frac{c_1}{\eta} = 0 \quad \text{for } \eta = 0 \text{ \& } \eta = 1$$

$$\theta = c_2 \text{ only!}$$

- We have "lost" another dimension! Now, we have a "0D" problem. This is called a lumped model.

  You can find the constant using a macroscopic balance. (more on this in a minute)

4. Differing time scales.

- Let's suppose we didn't drop the time derivative:

$$\rho \hat{C}_p \frac{\partial T}{\partial t} = \frac{k}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) + H_v$$

- The diffusion problem has an inherent timescale called the diffusion time (neglecting H_v for a moment)

$$\frac{\partial T}{\partial t} = \frac{\alpha}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) \qquad \text{let } \theta = \frac{T - T_\infty}{\Delta T}, \; \tilde{r} = r/R, \; \tilde{t} = t/t_d$$

diffusion time

$$\frac{\Delta T}{t_d} \frac{\partial \theta}{\partial \tilde{t}} = \frac{\alpha \Delta T}{R^2} \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}}\left(\tilde{r} \frac{\partial T}{\partial \tilde{r}}\right)$$

$$\frac{\Delta T}{t_d} \sim \frac{\alpha \Delta T}{R^2} \Rightarrow \boxed{t_d \sim R^2/\alpha} \; [=] \; \frac{\text{len}^2}{\text{len}^2/\text{time}} \; [=] \; \text{time}$$

- Now, suppose that H_v changes with some other time scale, t_p. This "process time" is set by external considerations (change in wire current).

  - Setting $\tilde{t} = t/t_p$ gives instead

$$\frac{\Delta T}{t_p} \frac{\partial \theta}{\partial \tilde{t}} = \frac{\Delta T}{t_d} \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}}\left(\tilde{r} \frac{\partial T}{\partial \tilde{r}}\right) + \frac{H_v(t)}{\rho \hat{C}_p}$$

now a function of time.

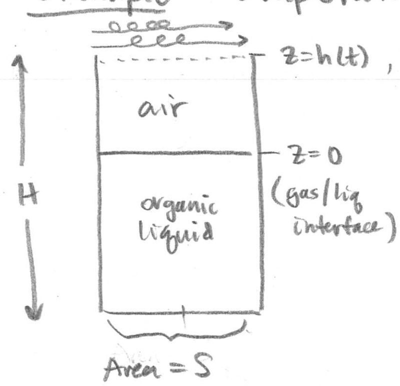
$$\frac{t_d}{t_p}\frac{\partial \theta}{\partial \tilde{t}} = \frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial T}{\partial \tilde{r}}\right) + \frac{t_d H_v}{\rho \tilde{c}_p}$$

- There are three possibilities:
  - (i) $t_d \ll t_p$: Diffusion is fast. This leads to a steady-state or pseudo steadystate problem.
  - (ii) $t_d \gg t_p$: Diffusion is slow. We will look at this in chapter 4. (similarity)
  - (iii) $t_d \sim t_p$: We have to solve the whole PDE, see ch. 5.

II. Pseudo-steady state problems.

* What happens if (like in the last example) we have some process with a slow timescale, but we still need to account for it?
  - We can solve to 1D transport Eq as if it is steady
  - We can use this to tell us information about the process time, if it is related. Usually this involves a macroscopic balance.

* Example: Evaporation of a liquid column.



Species A: air

Species B: organic vapor

x_A: stagnant & in equilibrium with the liquid

$x_B(z=0,t) = x_0$ (from vapor pressure)

$x_B(z=h(t),t) = 0$ (fast moving air)

* What is $h(t)$? Do a macroscopic balance on the liquid

$$\frac{d}{dt}\int_{V(t)} c_B \, dV = \int_{S(t)} \underline{N}_{Bz}\big|_{liquid} \cdot \underline{n} \, dS$$

- let $c_{B,L} = \frac{1}{V}\int c_B \, dV$, the average concentration

- Assume N_{Bz} is not a function of r.

$$\frac{d}{dt}\left(c_{B,L} V\right) = N_{Bz}\big|_{liquid} \cdot S$$

$V = S \cdot h(t)$

$c_{B,L}$ = constant.

(if V is big, it shouldn't change very much.)

$$\cancel{S}\, c_{B,L} \frac{dh}{dt} = N_{B,z}\big|_{liquid} \cancel{S}$$

$$\boxed{\frac{dh}{dt} = \frac{N_{B,z}\big|_{liq}}{c_{B,L}}}$$

- How get $N_{B,z}|_{liq}$? Use interface balance:

$$(\underline{F} - v_I)_{liq} = (\underline{F} - v_I)_{gas}$$

$$\boxed{N_{B,z}\big|_{liq} = N_{B,z,gas}} \quad *$$

* Moving reference frame of the interface. v_I is the same.

- what is the scale for t_p?

$$N_B \sim D_{AB}\frac{\partial c_B}{\partial z}$$

$$\frac{dh}{dt} = \frac{N_{B,z}|_{gas}}{c_{B,L}} \Rightarrow \frac{H}{t_p} \sim \frac{D_{AB}\, c_{B,G}/H}{c_{B,L}}$$

$$\boxed{t_p \sim \frac{H^2 c_{B,L}}{D_{AB}\, c_{B,G}}} \qquad \boxed{t_d \sim H^2/D_{AB}}$$

if $c_{B,L} >> c_{B,G}$ then we are pseudo steady.

* We can solve the species mass diffusion equation to find $c_B(z)$ and $N_{B,z}$. Use same methods as before. I'm out of time, but I'll post the notes as a supplement.