Lecture 13: The similarity method

I. Comments on solving PDEs

* We now move on to problems where we no longer have a single variable. We are now going to solve PDEs.

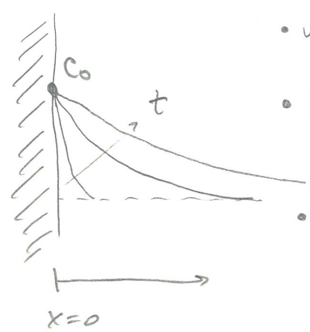
* This is not a PDE class per se, so we won't take the perspective of systematically solving PDEs in general, but we do need to know something about how to solve them.

* All analytical solutions to PDEs try to find some way to reduce them to one or more ODEs.

II. Diffusion in a semi-infinite medium

* We will use a method today called the "similarity method" or "combination of variables. I think it is easiest to show you an example first. Then we will pull what general lessons we can.

Example:



- what is $c(x,t)$?
- Concentration of species at wall diffusing into bulk fluid w/no c.
- For example, at the edge of a membrane.

B.C.'s: $c(0,t) = c_0$ I.C.: $c(x,0) = 0$

$c(\infty,t) = 0$

* Balance Equation :

$$\frac{Dc_i}{Dt} = D_i \nabla^2 c_i + R_{v,i}$$

dilute, constant ρ, constant D_i

- $\underline{v} = 0$ (dilute)
- $R_{v,i} = 0$
- constant in y, z

let $c_i = c$

$D_i = D$

$$\Rightarrow \quad \frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}$$

* This problem has an equivalent for heat transfer. Transient conduction in e.g. a solid.

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$T(0,t) = T_0$

$\left.\begin{array}{l} T(\infty, t) = T_a \\ T(x,0) = T_a \end{array}\right\}$ ambient temp.

(if $\theta = T - T_a$, get exactly the same Eq.)

* Let's non-dimensionalize this :

$$\theta = c/c_0 \qquad \tilde{x} = \frac{x}{L} \qquad \tilde{t} = t/\tau$$

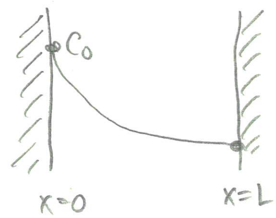
unspecified length and time scales for now.

$$\frac{c_0}{\tau} \frac{\partial \theta}{\partial \tilde{t}} = \frac{D \cdot c_0}{L^2} \frac{\partial^2 \theta}{\partial \tilde{x}^2}$$

$$\frac{\partial \theta}{\partial \tilde{t}} = \frac{D\tau}{L^2} \frac{\partial^2 \theta}{\partial \tilde{x}^2}$$

* what do we choose for L & τ ? let's consider a slightly easier case for the moment.

• What if we had a finite domain?



L comes from boundary conditions.

τ must be the difusion time:

$$\tau = L^2/D$$

• To see this, remember the Buckingham Pi Theorem:

	finite domain	semi-infinite domain
# dimensions	3 (len, mass, time)	3 (same)
# variables & constants	6 (c, t, x, D, C_0, L)	(*) 5 (c, t, x, D, C_0) $L \to \infty$!
# dimensionless groups	$6-3=3$ $\theta = \frac{c}{C_0}$, $\tilde{x} = \frac{x}{L}$, $\tilde{t} = tD/L^2$	$5-3=2$ $\theta = \frac{c}{C_0}$, $\tilde{x} = ?$ $\tilde{t} = ?$

* In a semi-infinite domain, we don't have enough degrees of freedom to have a separate group for x & t!

(*) Not an obvious jump. Took smart people a long time to figure out.

* There is a new, time-dependent length scale. We call it the penetration depth.

$$\text{penetration depth} \sim \sqrt{Dt} \quad [=] \left(\frac{\text{Len}^2}{\text{time}} \cdot \text{time}\right)^{1/2}$$

* Physical interpretation: length scale that characterizes how far concentration has penetrated into the medium.

* Concentration reaches depth L at the diffusion time, t_d.

$$L \sim \sqrt{Dt} \quad \Rightarrow \quad t \sim L^2/D$$

III. The similarity Method.

* The idea of the similarity method is to combine the variables x & t using the penetration depth:

$$\eta = \frac{x}{g(t)} \qquad g(t) = \sqrt{aDt}$$

some unspecified constant.

* Let's rewrite our balance equation & boundary conditions using this idea.

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} \qquad c(0,t) = c_0 \qquad c(x,0) = 0$$
$$c(\infty,t) = 0$$

$$\theta = \frac{c}{c_0} \qquad \eta = \frac{x}{\sqrt{aDt}}$$ Go term by term.

- $$\frac{\partial c}{\partial t} = \left(\frac{\partial c}{\partial t}\right)_x = \frac{dc}{d\eta}\left(\frac{\partial \eta}{\partial t}\right)_x = c_0 \frac{d\theta}{d\eta} \frac{\partial}{\partial t}\left(\frac{x}{\sqrt{aDt}}\right)_x$$

chain rule

$$= c_0 \frac{d\theta}{d\eta}\frac{\partial}{\partial t}\left[x(aDt)^{-1/2}\right]_x$$

$$= c_0 \cdot \frac{d\theta}{d\eta}\left[-\frac{xaD}{2}(aDt)^{-3/2}\right]$$

$$= -\frac{c_0}{2}\frac{d\theta}{d\eta}\left[\underbrace{\frac{x}{(aDt)^{1/2}}}_{\eta} \cdot \frac{\cancel{a}D}{\cancel{a}Dt}\right]$$

$$= -\frac{c_0}{2}\frac{\eta}{t}\frac{d\theta}{d\eta}$$

chain rule

- $$\frac{\partial c}{\partial x} = \left(\frac{\partial c}{\partial x}\right)_t = \frac{dc}{d\eta}\left(\frac{\partial \eta}{\partial x}\right)_t = c_0\frac{d\theta}{d\eta}\cdot\frac{\partial}{\partial x}\left[\frac{x}{\sqrt{aDt}}\right]_t$$

need for 2nd derivative.

$$= c_0 \frac{d\theta}{d\eta}\frac{1}{\sqrt{aDt}}$$

- $$\frac{\partial^2 c}{\partial x^2} = \left[\frac{\partial}{\partial x}\left(\frac{\partial c}{\partial x}\right)\right]_t = \left[\frac{d}{d\eta}\left(\frac{\partial c}{\partial x}\right)\frac{\partial \eta}{\partial x}\right]_t$$

above $\quad$ also had above

$c_0 \frac{d\theta}{d\eta}\frac{1}{\sqrt{aDt}} \qquad \frac{1}{\sqrt{aDt}}$

$$= \left[\frac{d}{d\eta}\left(c_0 \frac{d\theta}{d\eta}\frac{1}{\sqrt{aDt}}\right)\frac{1}{\sqrt{aDt}}\right]_t = \frac{c_0}{aDt}\frac{d^2\theta}{d\eta^2}$$

- Plug into differential equation

$$-\frac{\cancel{c_0}}{2}\frac{\eta}{\cancel{t}}\frac{d\theta}{d\eta} = \cancel{D}\frac{\cancel{c_0}}{a\cancel{D}\cancel{t}}\frac{d^2\theta}{d\eta^2}$$

$$\Rightarrow \boxed{\frac{d^2\theta}{d\eta^2} + \frac{a\eta}{2}\frac{d\theta}{d\eta} = 0}$$

- we also need to transform the boundary conditions & initial condition

BC1: $c(x=0,t) = c_0 \qquad \eta = \frac{x}{\sqrt{aDt}} = \frac{0}{\sqrt{aDt}} = 0 \quad , \theta = c_0/c_0$

$$\Rightarrow \boxed{\theta(0) = 1}$$

BC2: $c(x=\infty,t) = 0 \qquad \eta = \frac{\infty}{\sqrt{aDt}} = \infty \quad , \theta = 0/c_0$

$$\Rightarrow \boxed{\theta(\infty) = 0}$$

I.C. : $c(x,t=0) = 0 \qquad \eta = \frac{x}{\sqrt{aD\cdot 0}} = \infty \quad , \theta = 0/c_0$

$$\Rightarrow \boxed{\theta(\infty) = 0}$$

- Phew, BC2 & the IC became the same condition. This is needed because we now have an ODE & can only have 2 BCs.

* Now, let's solve the ODE.

$$\frac{d^2\theta}{d\eta^2} + \frac{a\eta}{2}\frac{d\theta}{d\eta} = 0 \qquad \theta(0)=1 \;,\; \theta(\infty)=0$$

- trick, define $\psi = \frac{d\theta}{d\eta}$

$$\frac{d\psi}{d\eta} + \frac{a\eta}{2}\psi = 0 \quad \leftarrow \text{separate \& integrate}$$

$$\frac{1}{\psi}\frac{d\psi}{d\eta} = -\frac{a\eta}{2} \quad \Rightarrow \quad \ln\psi = -\frac{a\eta^2}{4} + k_1'$$

$$\psi = k_1 \exp(-a\eta^2/4)$$

$$\frac{d\theta}{d\eta} = k_1 \exp(-a\eta^2/4) \quad \leftarrow \text{integrate again}$$

$$\boxed{\theta = k_1 \int_0^\eta \exp(-as^2/4)\, ds + k_2}$$

dummy variable of integration

* Apply BC's to solve for k_1 & k_2:

$$\theta(0) = k_1 \int_0^0 \cancel{\exp(-as^2/4)\, ds} + k_2 = 1$$

$$\boxed{k_2 = 1}$$

$$\theta(\infty) = k_1 \underbrace{\int_0^\infty \exp(-as^2/4)\, ds}_{\text{known! Gaussian Integral : } \sqrt{\pi/a} \text{ for } a>0 \text{ \& real.}} + 1 = 0$$

$$= k_1\sqrt{\pi/a} + 1 = 0$$

$$\Rightarrow \boxed{k_1 = -\sqrt{a/\pi}}$$

* Combine them:

$$\theta(\eta) = 1 - \sqrt{\frac{a}{\pi}} \int_0^\eta \exp(-as^2/4)\, ds$$

- we left a unspecified. we pick it to be $a=4$.
  we do this for convenience to get rid of fraction.

  ↳ $\eta = \dfrac{x}{\sqrt{4Dt}}$

$$\boxed{\theta(\eta) = 1 - \underbrace{\frac{2}{\sqrt{\pi}} \int_0^{\eta} \exp(-s^2)\, ds}_{}}$$

This is a special function called the error function, $\mathrm{erf}(\eta)$.

$$\boxed{\theta = 1 - \mathrm{erf}(\eta) = \mathrm{erfc}(\eta)}$$

another special function. "complimentary error function."

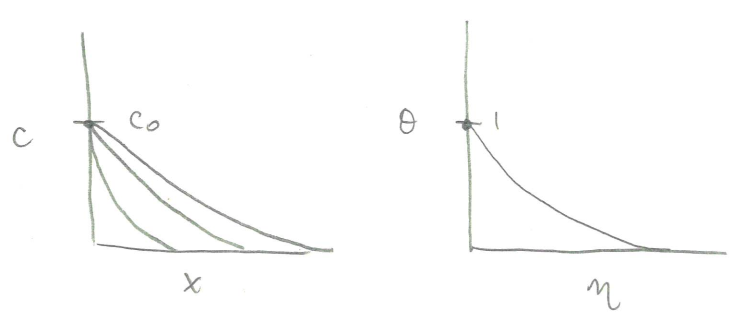
- erf & erfc are widely tabulated, just like sin, cos, exp, etc.

IV. Comments

* what did we just do? we found out that if we combined variables: $\eta = x/2\sqrt{Dt}$ we got an ODE instead of a PDE.

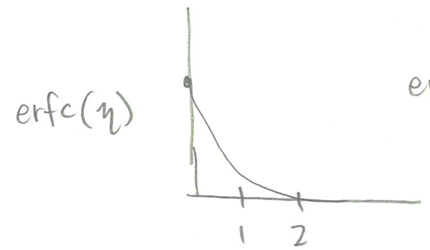
why did this work? we actually only had one independent variable (η) not two (x & t).

* Another way to see this is a plot. When properly scaled, the solutions all have the same shape! We say the solutions are self-similar. The transformation $x/g(t)$ is called a similarity transform.



* This only works when the domain has no natural length scale (i.e. is infinite).

* $g(t)$ is a penetration depth. It is how far our species has diffused into the domain at time t.



$\mathrm{erfc}(2) \approx 0.005$

So at $\eta = 2$, less than 1% of the species concentration has penetrated the bulk.

* Deen has a more general analysis where he doesn't assume a form for $g(t)$. In this case you substitute: $\eta = x/g(t)$. (see example 4.2-1)

The chain rule gives:

$$\frac{d^2\theta}{d\eta^2} + \frac{gg'}{D}\eta\frac{d\theta}{d\eta} = 0 \qquad g' = \frac{dg}{dt}$$

For the similarity transform to give η independent of t,

$$gg' = \text{const}$$

This allows us to solve for $g(t)$:

$$\frac{g}{D}\frac{dg}{dt} = c_1 \Rightarrow \int g\,dg = \int c_1 D\,dt$$

$$\Rightarrow \tfrac{1}{2}g^2 = c_1 D t + c_2$$

Let $g(0) = 0 \Rightarrow c_2 = 0 \rightarrow$ penetration depth = 0 at time = 0.

$$\Rightarrow g = \sqrt{2c_1 D t}$$

if $c_1 = 2$, we get same as above

$$g = \sqrt{4Dt}$$