Lecture 14 – The FFT method for solving PDEs

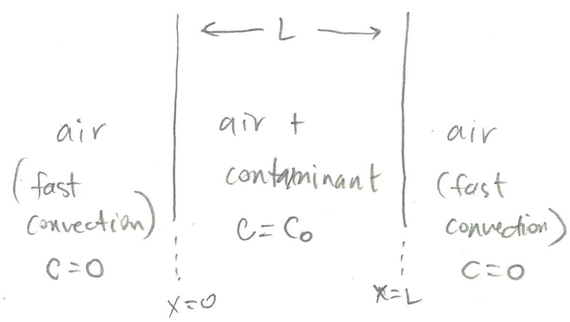
* Last time we learned about the function – vector analogy, and that we can represent functions as Fourier series:

$$f(x) = \sum_{n=1}^{\infty} c_n \, \varphi_n(x) \qquad \text{Fourier Series or Inverse FFT.}$$

$$c_n = \int_a^b f(x)\, \varphi_n(x)\, dx \qquad \text{Finite Fourier Transform}$$

* Today we are going to use this fact to solve linear PDEs.

I. Example : Transient Diffusion (c.f. Example 5.3-2 in Deen)



* Balance Equation:

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} \qquad \begin{aligned} c(x,0) &= C_0 \\ c(0,t) &= 0 \\ c(L,t) &= 0 \end{aligned}$$

* Non-Dimensionalize:

$$\theta = c/c_0 \,, \quad \tilde{x} = x/L \,, \quad \tilde{t} = tD/L^2$$

drop tildes. $\longrightarrow$

$$\cancel{\frac{C_0 D}{L^2}} \frac{\partial \theta}{\partial \tilde{t}} = \cancel{\frac{C_0 D}{L^2}} \frac{\partial^2 \theta}{\partial \tilde{x}^2} \qquad \begin{aligned} c(\tilde{x},0) &= 1 \\ c(0,\tilde{t}) &= 0 \\ c(1,\tilde{t}) &= 0 \end{aligned}$$

* Math problem:

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} \qquad c(x,0) = 1 \,, \quad \text{initial condition}$$

$$c(0,t) = 0 \,, \; c(1,t) = 0 \,, \quad \text{boundary conditions.}$$

(1) Identify Basis functions

- The basis functions correspond to the eigenvalue problem for dimension with homogeneous boundary conditions.

- In our case we have two dimensions: t, x.
  - t has an inital condition
  - x has homogeneous boundary conditions *
- Differential operator for x: $\mathcal{L} = \frac{\partial^2}{\partial x^2}$ with homogeneous Dirichlet conditions
- what are eigenfunctions for

$$\mathcal{L}\psi = \lambda\psi \;?$$

we found them last time: $\psi(x) = \sqrt{2}\,\sin(n\pi x)$

- we will talk more about this for other operators & problems later in the lecture.

(2) Now, we define the FFT and take the FFT of our PDE

$$\underbrace{c_n(t)}_{\text{the coefficient}} = \int_0^1 \underbrace{\theta(x,t)}_{\text{The function we seek}}\;\underbrace{\sqrt{2}\,\sin(n\pi x)}_{\psi_n(x)}\;dx$$

dx ↑ the dimension of the eigenvalue problem

- Take FFT of our PDE

$$\underbrace{\int_0^1 \sqrt{2}\,\sin(n\pi x)\,\frac{\partial\theta}{\partial t}\,dx}_{\text{LHS}} = \underbrace{\int_0^1 \sqrt{2}\,\sin(n\pi x)\,\frac{\partial^2\theta}{\partial x^2}\,dx}_{\text{RHS}}$$

LHS: $$\int_0^1 \sqrt{2}\,\sin(n\pi x)\,\frac{\partial\theta}{\partial t}\,dx = \frac{d}{dt}\underbrace{\int_0^1 \sqrt{2}\,\sin(n\pi x)\,\theta\,dx}_{c_n(t)}$$

$$= \frac{dc_n}{dt}$$

RHS: $$\int_0^1 \sqrt{2}\,\sin(n\pi x)\,\frac{\partial^2\theta}{\partial x^2}\,dx = \left[\sqrt{2}\,\sin(n\pi x)\,\frac{\partial\theta}{\partial x}\right]_0^1 - \int_0^1 \frac{\partial\theta}{\partial x}\,\sqrt{2}\,n\pi\,\cos(n\pi x)\,dx$$

integration by parts

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

- $\sin(n\pi)=0$ & $\sin(0)=0$, so the first term is zero

$$= -\int_0^1 \frac{\partial\theta}{\partial x}\sqrt{2}\,n\pi\cos(n\pi x)\,dx = -\left[\sqrt{2}\,n\pi\cos(n\pi x)\,\theta\right]\Big|_0^1 + \int_0^1 \theta\cdot\sqrt{2}\,n^2\pi^2(-1)\sin(n\pi x)\,dx$$

I.B.P again

- $\theta(0)=0$ & $\theta(1)=0$ from boundary conditions

$$= -(n\pi)^2 \underbrace{\int_0^1 \sqrt{2}\sin(n\pi x)\,\theta\,dx}_{c_n} = -(n\pi)^2 c_n$$

- Put together LHS & RHS

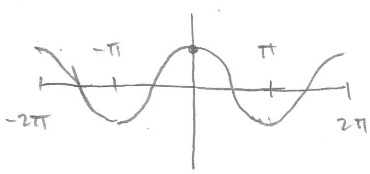
$$\frac{dc_n}{dt} = -(n\pi)^2 c_n$$

Our transform converted the PDE into an ODE!
we got rid of the x-dimension.

- we also need to transform the initial condition:

$$c(x,0)=1 \rightarrow \int_0^1 \overset{1}{\cancel{c(x,0)}}\cdot\sqrt{2}\sin(n\pi x)\,dx$$

$$= \int_0^1 \sqrt{2}\sin(n\pi x)\,dx = -\frac{\sqrt{2}}{n\pi}\cos(n\pi x)\Big|_0^1$$



if $x=0$, $\cos(0)=1$
if $x=1$, $\cos(n\pi)=(-1)^n$

$$= -\frac{\sqrt{2}}{n\pi}\left((-1)^n-1\right) = \frac{\sqrt{2}}{n\pi}\left(1-(-1)^n\right), \quad n=1,2,\dots$$

$$c_n(0) = \begin{cases} \dfrac{2\sqrt{2}}{n\pi}, & n=1,3,5,\dots \text{ odds} \\ 0, & n=0,2,\dots, \text{ evens} \end{cases}$$

(3) Now, solve the transformed equation

$$\frac{dc_n}{dt} = -(n\pi)^2 c_n \Rightarrow \text{separable, integrable}$$

$$c_n(t) = a_n \exp(-n^2\pi^2 t)$$

- apply initial condition

$$c_n(0) = a_n = \frac{\sqrt{2}}{n\pi}\left[1-(-1)^n\right]$$

$$c_n(t) = \frac{\sqrt{2}}{n\pi}\left[1-(-1)^n\right] \exp(-n^2\pi^2 t)$$

(4) Take the "inverse" FFT, i.e. use Fourier Series

$$f(x) = \sum_{n=1}^{\infty} c_n(t)\, \varphi_n(x)$$

$$= \sum_{n=1}^{\infty} \frac{\sqrt{2}}{n\pi}\left[1-(-1)^n\right] \exp(-n^2\pi^2 t) \cdot \sqrt{2} \sin(n\pi x)$$

simplify :

$$\boxed{f(x) = \sum_{\substack{n=1,3,5\ldots \\ \text{(odd)}}}^{\infty} \frac{4}{n\pi} \exp(-n^2\pi^2 t) \sin(n\pi x)}$$

we did it! Solved a PDE!

* Comments

- See plot in Python. Is it what you expected?
- Related to "separation of variables." Note that x & t are separately solved.
- 4 steps :
  (1) Identify Basis Functions for FFT.
  (2) Take FFT of PDE
  (3) Solve the transformed Eq.
  (4) take inverse FFT.

II. Variations to FFT Problems

* There are two different ways that FFT problems / linear PDEs vary.
  - Basis Functions
  - Inhomogeneities & steady states

A. Basis Functions

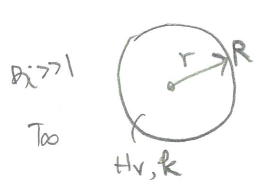
* Basis functions are determined by the dimension of the eigenvalue problem (the one with homogeneous boundary conditions) and the boundary conditions. (Remember the BCs change the operator!)

* Deen has a lot of examples for cartesian, cylindrical, & spherical coordinates. He summarizes them all in Table 5-3. Details are given in:

  Table 5-2 - Cartesian
  Table 5-4 - Cylindrical
  Table 5-5 & 5-6 - Spherical

* You should be able to derive them if need be, but the tables make it easy to use them!

* Example: Transient conduction in a heated wire (Like Example 5.7-1, but w/ Dirichlet conditions)



$\theta = \frac{T - T_\infty}{H_V R^2/k}$

$\tilde{r} = r/R$

$\tilde{t} = tD/R^2$

$$\frac{\partial \theta}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) + 1$$

$\theta(r,0) = 0$

$\theta(0,t) = 0 \quad , \quad \theta(1,t) = 0$

- what direction is the eigenvalue problem?

The one with homogeneous BCs. $\rightarrow$ r.

$$\mathcal{L}\psi = \lambda\psi \quad \Rightarrow \quad \frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = \lambda\psi \, , \quad \begin{matrix}\psi(0)=0 \\ \psi(1)=0\end{matrix}$$

(why not the z? That is inhomogeneous PDE. S in $\mathcal{L}\theta = S$)

- let's solve:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) - \lambda\psi = 0$$

"cylindrical harmonics"

Appendix B $\rightarrow$ if $\lambda = -m^2$ then this is Bessel's equation of order 0.

$$\psi(r) = a\,J_0(mr) + b\,Y_0(mr)$$

J_i: 1st kind
Y_i: 2nd kind

* The differential operator determines the fundamental solution. The BCs specify the basis function.

- Apply the BCs:

$$\psi(0) = 0 \Rightarrow Y_0(0) = -\infty \, , \text{ so } b = 0$$

$$\psi(1) = a\,J_0(m) = 0 \quad \Rightarrow \text{ if } a=0 \text{, only trivial solution, so we need zeros of } J_0! \text{ call them } m_n.$$

$$\boxed{\psi(r) = a_n\,J_0(m_n r)}$$

- Now, normalize to find a_n:

$$(\psi,\psi) = 1 = \int_0^1 a_n^2\,J_0^2(m_n r)\, r\,dr$$

App B Eq. B.4-6.

There is an error in this Eq. in Book

$$1 = a_n^2\left[\frac{1}{2}\left(J_0^2(m_n) + J_1^2(m_n)\right)\right]$$

(*) $w(r) = r$ is the weighting function for this inner product.

$$a_n = \left[\frac{2}{J_0^2(m_n) + J_1^2(m_n)} \right]^{1/2} = \frac{\sqrt{2}}{J_1(m_n)}$$

zeros !

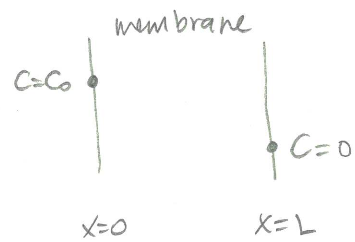
$$\boxed{\psi_n(r) = \frac{\sqrt{2}\, J_0(m_n r)}{J_1(m_n)}}$$

* Now, use this basis function in Fourier Series

B. Inhomogeneities and Steady States

* Another complication arises when there are multiple inhomogeneous boundary conditions or steady states.

* Example: Transient diffusion with a steady state.



* $\dfrac{\partial c}{\partial t} = D \dfrac{\partial^2 c}{\partial x^2}$ $\quad c(0,t) = c_o$

$c(L,t) = 0$

$c(x,0) = 0$

* $\theta = c/c_o, \quad \tilde{x} = x/L, \quad \tilde{t} = tD/L^2$

drop tildes

* Dimensionless Balance:

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} \qquad \theta(0,t) = 1$$

$$\theta(1,t) = 0$$

$$\theta(x,0) = 0$$

* How do we solve this one? There is no dimension with homogeneous boundary conditions: $t \rightarrow$ initial condition

$x \rightarrow$ inhomogeneous.

* There is a steady state:

$$\frac{d^2\theta_{ss}}{dx^2} = 0 \qquad \theta_{ss}(0) = 1, \quad \theta_{ss}(1) = 0$$

integrate 2x

$$\theta_{ss} = ax + b \qquad \theta_{ss}(0) = 1 = b, \quad \theta_{ss}(1) = a + b = 0$$

$$\theta_{ss} = 1 - x \qquad a = -b = -1$$

* Split the problem into two. This will work because of superposition.

$$\theta(x,t) = \theta_{ss}(x) + \theta_t(x,t)$$

* Now, re-write our balance for θ_t:

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} \Rightarrow \frac{\partial(\theta_{ss}+\theta_t)}{\partial t} = \frac{\partial^2(\theta_{ss}+\theta_t)}{\partial x^2}$$

$$\Rightarrow \underbrace{\cancel{\frac{\partial \theta_{ss}}{\partial t}}}_{0} + \frac{\partial \theta_t}{\partial t} = \underbrace{\cancel{\frac{\partial^2 \theta_{ss}}{\partial x^2}}}_{0} + \frac{\partial^2 \theta_t}{\partial x^2}$$

$$\boxed{\frac{\partial \theta_t}{\partial t} = \frac{\partial^2 \theta_t}{\partial x^2}}$$

* Obey's same PDE.
(It should because of superposition

* Also the Boundary/Initial conditions

$$\theta_t = \theta - \theta_{ss} \Rightarrow \theta_t(0,t) = \theta(0,t) - \theta_{ss}(0,t) = 1 - 1 = 0$$

$$\theta_t(1,t) = \theta(1,t) - \theta_{ss}(1,t) = 0 - 0 = 0$$

$$\theta_t(x,0) = \theta(x,0) - \theta_{ss}(x) = 0 - (1-x) = x-1$$

$$\boxed{\begin{aligned} \theta_t(0,t) &= 0 \\ \theta_t(1,t) &= 0 \\ \theta_t(x,0) &= x-1 \end{aligned}}$$

* Swapped an inhomogeneous boundary condition for an inhomogeneous initial condition.

* The problem proceeds the same from here. the same as before. The only difference is the initial condition; we will need a FFT of it

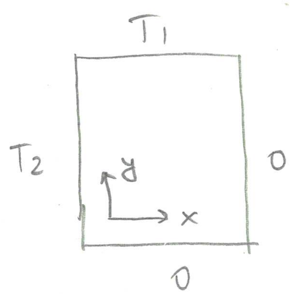
* Details are in Example 5.6-2 in Deen.

$$\theta_t(x,t) = -\sum_{n=1}^{\infty} \frac{2\sin(n\pi x)}{n\pi} \exp(-n^2\pi^2 t)$$

$$\theta(x,t) = 1 - x - 2\sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n\pi} \exp(-n^2\pi^2 t)$$

* Take away / General Stategy: when you have inconvenient inhomogeneities, write a sum so you can divide or re-arrange those inhomogeneities. Superposition will mean they obey the same PDE & you can add them up at the end.

* Another example:



$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

2 sides are inhomogeneous?

$T = T_a + T_b$

T_a: has x inhomogeneous BC

T_b: " y " BC.