Lecture 18 - Inviscid & Irrotational Flow

I. Overview of Inviscid Flow

* In contrast to creeping flow, inviscid flow occurs at high Reynolds number:

$$\rho \frac{D\underline{v}}{Dt} = -\underline{\nabla}P$$

or

$$\rho \underline{v}\cdot\underline{\nabla}\underline{v} = -\underline{\nabla}P \quad \text{(steady)}$$

* Inviscid Flow is also very relevant for engineers. Applications include:
    - Airplanes, cars, parachutes, bikes, etc.
    - Jets, i.e. nozzles, burners, etc.
    - stirred tanks

* These are things that are big, low viscosity, and move fast.

* Inviscid flow is more intuitive to you. It has inertia. It is not reversible. You already have spent a lot of time considering & watching inviscid flows

* Inviscid flows obey Bernoulli's equation:

$$\underline{v}\cdot\underline{\nabla}\left(\frac{v^2}{2} + \frac{P}{\rho} + gh\right) = 0$$

→ mechanical energy is constant along a streamline

- Reversible interconversion of mechanical energy because there are no friction losses.

* Mathematically, Inviscid Flow is not as simple as creeping flow.

- It is non-linear.
- Multiple solutions are possible. Indeed we know turbulence exists at high Re.
- No more superposition

* However, there is something we can still do! Recall the vorticity transport equation:

$$\frac{D\underline{\omega}}{Dt} = \underline{\omega} \cdot \underline{\nabla}\, \underline{v} + \nu \nabla^2 \underline{\omega}$$

- At high Re, this becomes

$$\frac{D\underline{\omega}}{Dt} = \underline{\omega} \cdot \underline{\nabla}\, \underline{v} \qquad \text{(inviscid)}$$

- At steady state, it becomes

$$\underline{v} \cdot \underline{\nabla}\, \underline{\omega} = \underline{\omega} \cdot \underline{\nabla}\, \underline{v}$$

- Now, suppose that we have a flow that is uniform somewhere in the domain

$v_x = U$

- $\underline{\nabla}\,\underline{v} = \underline{0}$ in the uniform region

$$\Rightarrow \quad \underline{v} \cdot \underline{\nabla}\, \underline{\omega} = \underline{0}$$

not zero

rate of change of $\underline{\omega}$ is zero.

- But $\underline{\omega} = \underline{0}$ also in the uniform region!
  $(\underline{\nabla} \times \underline{v} = 0)$
- If $\underline{\omega} = \underline{0}$ & the rate of change of $\underline{\omega}$ is zero, then ω must stay zero!

- Key Idea: If an inviscid flow is irrotational ($\underline{\omega} = \underline{0}$) somewhere, the whole streamline remains irrotational.

- Now, we don't need our complex vorticity equation, we only need $\boxed{\underline{\omega} = \underline{0}}$! More on math in a minute

* what did we just do? we found a subset of inviscid flows that are more simple. These are irrotational flows. They are the laminar flows over objects or in jets or in stirred tanks. Turbulent flows have significant viscosity!

  Aside: For reasons we will soon encounter, one cannot truly neglect viscosity (Boundary layers!). Because of this vorticity is "generated" near surfaces, leading to turbulence.

* why do this?
  - We can make analytical progress. Helps us understand more complex cases.
  - Case in point: Boundary layers. It will be a big deal for understanding them.
  - History - it's what folks did first in Fluid Mech!

* <u>Focusing on Irrotational Flow</u>, how can we mathematically describe them? There are two ways:

- stream function: $\nabla^2 \psi = 0$ (planar)

  $E^2 \psi = 0$ (axisymmetric)

  - remember these from creeping flow? Now only 2^{nd} order because $\underline{\omega} = 0$.

- velocity potential:
  - Instead of using the streamfunction, one can define a velocity potential, ϕ:

    $$\underline{v} = \underline{\nabla} \phi$$

  - why? All vector fields can be written as a sum: $\underline{v} = \underbrace{\underline{\nabla}\phi}_{\text{curl-free}} + \underbrace{\underline{\nabla} \times \underline{A}}_{\text{divergence free}}$

    (Helmholtz Decomposition).

  - why #2? Analogy with forces for systems that are conservative (like thermo)

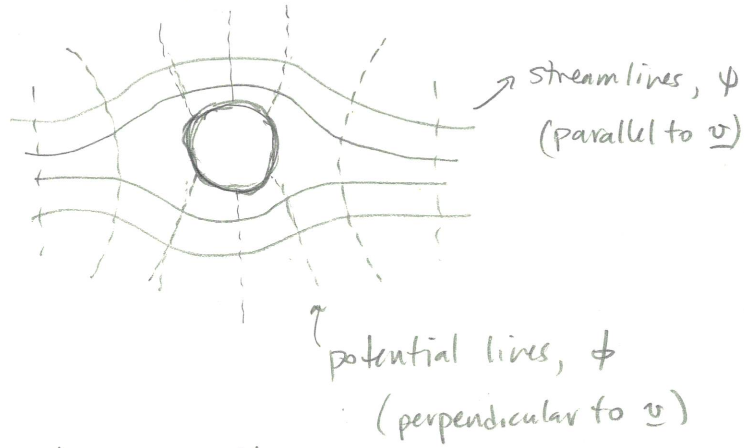
    $$\underline{F} = -\underline{\nabla} u \leftarrow \text{potential energy}$$

    $$\underline{v} = \nabla \phi$$

  - Since we know that $\underline{\nabla} \cdot \underline{u} = 0$, we can write: $\underline{\nabla} \cdot (\underline{\nabla} \phi) = \boxed{\nabla^2 \phi = 0}$

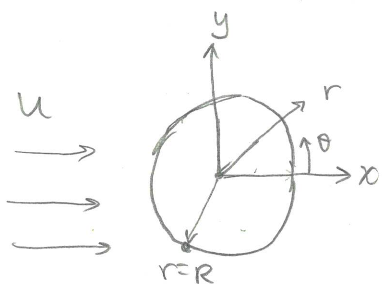
  - This is also 2^{nd} order PDE. But it is good in <u>3D</u>! No E^2 in curvilinear coords!

- what does ϕ mean? Like potential lines in E & M:



- Irrotational flow is often called "potential flow" or "ideal flow".

II. Example: Irrotational Flow Past a Cylinder



z - into the paper

- cylindrical coordinates, z - not of interest

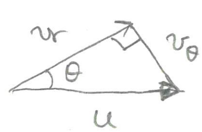
$$\nabla^2 \psi = 0, \quad v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad v_\theta = -\frac{\partial \psi}{\partial r}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

* Boundary conditions:

$$v_r(R,\theta) = 0 \longrightarrow \frac{\partial \psi}{\partial \theta}(R,\theta) = 0 \quad \checkmark$$

$$v_\theta(R,\theta) = 0 \longrightarrow \frac{\partial \psi}{\partial r}(R,\theta) = 0 \quad \checkmark$$



$$v_r(r\to\infty,\theta) = U\cos\theta$$

$$v_\theta(r\to\infty,\theta) = -U\sin\theta$$

Integrating:

$$\left.\begin{aligned} \int r v_r \, d\theta &= rU\sin\theta \\ -\int v_\theta \, dr &= rU\sin\theta \end{aligned}\right\} \psi(r\to\infty,\theta) = rU\sin\theta \quad \checkmark$$

* Now, solve via FFT method.

$$C_n(r) = \int_a^b f(r,\theta)\, D_n(\theta)\, w(\theta)\, d\theta \qquad (\star)$$

$w(\theta) = 1$

eigenfunctions (use D_n to avoid ψ_n)

$$f(r,\theta) = \sum_{n=1}^{\infty} C_n(r)\, D_n(\theta) \qquad (\star\star)$$

- page 190, eigen functions are sin & cos. (§ 5.7)

$$D_n(\theta) = \begin{cases} 1/\sqrt{2\pi} & , n=0 \\ \frac{1}{\sqrt{\pi}} \sin \frac{(n+1)\theta}{2} & , n=1,3,5,\dots \\ \frac{1}{\sqrt{\pi}} \cos\left(\frac{n\theta}{2}\right) & , n=2,4,6,\dots \end{cases}$$

This is a pain, because we have an odd-even effect. Luckily our problem is relatively simple.

- Normally, our next step in the FFT method would be to transform the BVP using $(\star)$. However note that the boundary conditions for $\psi(r,\theta)$ must satisfy $\psi \sim \sin\theta$.

This is an eigenfunction $(n=1)$! So, our solution is already "along" an eigen function. We don't need the infinite sum in $(\star\star)$. we only need $\sin(\theta)$.

$\Rightarrow$ let $\psi(r,\theta) = c(r)\sin\theta$.

- Substitute this into our PDE

$$\nabla^2\psi = 0 \quad \Rightarrow \quad \nabla^2(c\sin\theta) = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial(c\sin\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2(c\sin\theta)}{\partial\theta^2} = 0$$

$$\cancel{\sin\theta}\;\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right)}_{\text{expand}} - \frac{c}{r^2}\,\cancel{\sin\theta} = 0$$

$$\frac{1}{r}\,r\,\frac{\partial^2 c}{\partial r^2} + \frac{1}{r}\frac{\partial c}{\partial r} - \frac{c}{r^2} = 0$$

$$\frac{\partial^2 c}{\partial r^2} + \frac{1}{r}\frac{\partial c}{\partial r} - \frac{c}{r^2} = 0$$

This is the same thing we would get from an FFT transform.

- This is a equidimensional equation.
  It has the form of solution:

$$c(r) = r^m$$

(kind of like e^r or sins/cos with const. coeff. equations)

(notice all terms match derivatives = equidimensional)

- Substitute:

$$(m-1)m\,r^{m-2} + \frac{1}{r}\cdot m\,r^{m-1} - \frac{1}{r^2}r^m = 0$$

$$(m-1)m + m - 1 = 0$$

$$m^2 - m + m = 1 \;\Rightarrow\; m^2 = 1 \;\Rightarrow\; m = \pm 1$$

$$c(r) = Ar + Br^{-1}$$

- Put all together and apply BC's:

$$\psi(r,\theta) = \left(Ar + Br^{-1}\right)\sin\theta$$

$$\psi(R,\theta) = 0 = \left(AR + B/R\right)\sin\theta \;\Rightarrow\; AR + B/R = 0$$

$$B = -AR^2$$

$$\lim_{r\to\infty}\psi(r,\theta) = Ar\sin\theta = Ur\sin\theta \;\Rightarrow\; A = U$$

$$\psi(r,\theta) = \left(Ur - UR^2/r\right)\sin\theta$$

$$\boxed{\psi(r,\theta) = UR\sin\theta\left(\frac{r}{R} - \frac{R}{r}\right)}$$

- Now, in terms of velocity

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{UR}{r}\cos\theta\left(\frac{r}{R} - R/r\right) = U\cos\theta\left(1 - \frac{R^2}{r^2}\right)$$

$$v_\theta = -\frac{\partial\psi}{\partial r} = -UR\sin\theta\left(\frac{1}{R} + \frac{R}{r^2}\right) = -U\sin\theta\left(1 + \frac{R^2}{r^2}\right)$$

$$\boxed{\begin{aligned} v_r &= U\cos\theta\left(1 - \frac{R^2}{r^2}\right) \\ v_\theta &= -U\sin\theta\left(1 + R^2/r^2\right) \end{aligned}}$$

* Comments

- A bit complex because we have non-constant BC's in cylindrical coordinates. It ended up working out though.
- Notice something weird:

$$v_\theta(r=R) = -U\sin\theta\left(1 + R^2/R^2\right) = -2U\sin\theta$$

$$v_r(r=R) = U\cos\theta\left(1 - R^2/R^2\right) = 0 \quad \checkmark \text{ ok.}$$

Why is v_θ not 0 at $r=R$?

→ we didn't use it / need it.

Something weird is going on. Don't satisfy no slip.

"D'Alembert's Paradox."

Messy FFT Derivation of cylinder potential flow

$$\int \left[r \frac{d}{dr}\left(r \frac{d}{dr}\psi\right) + \frac{d^2\psi}{d\theta^2} \right] \frac{1}{\sqrt{\pi}} \sin\left(\frac{(n+1)\theta}{2}\right) d\theta$$

$$\int \left[r \frac{d}{dr}\left(r \frac{d\psi}{dr}\right)\right] \frac{1}{\sqrt{\pi}} \sin\left(\frac{(n+1)\theta}{2}\right) d\theta + \int_0^{2\pi} \frac{d^2\psi}{d\theta^2} \frac{1}{\sqrt{\pi}} \sin\left(\frac{(n+1)\theta}{2}\right) d\theta$$

IBP.

$u = 1/\sqrt{\pi} \sin\left(\frac{(n+1)\theta}{2}\right)$

$du = \frac{(n+1)}{2} \frac{1}{\sqrt{\pi}} \cos\left(\frac{(n+1)\theta}{2}\right)$

$dv = \frac{d^2\psi}{d\theta^2}$

$v = \frac{d\psi}{d\theta}$

$$\cancel{\frac{d\psi}{d\theta} \cdot \frac{1}{\sqrt{\pi}} \sin\left(\frac{(n+1)\theta}{2}\right)\Bigg|_0^{2\pi}}^{\,0} - \int \frac{(n+1)}{2} \frac{1}{\sqrt{\pi}} \cos\left(\left(\frac{n+1}{2}\right)\theta\right) \cdot \frac{d\psi}{d\theta}\, d\theta$$

$\sin 0 = 0$

$\sin\left(\frac{(n+1)\cdot \cancel{2}\pi}{\cancel{2}}\right)$

$\sin(n\pi) = 0$

$u = \cos\left(\left(\frac{n+1}{2}\right)\theta\right) \cdot \frac{(n+1)}{2\sqrt{\pi}}$

$du = \frac{n+1}{2} \sin\left(\frac{n+1}{2}\theta\right) \cdot \frac{(n+1)}{2\sqrt{\pi}}$

$dv = \frac{d\psi}{d\theta} \quad v = \psi$

$$\Rightarrow \quad - \frac{n+1}{2\sqrt{\pi}} \psi \cos\left(\left(\frac{n+1}{2}\right)\theta\right)\Bigg|_0^{2\pi} + \int_0^{2\pi} \left(\frac{n+1}{2}\right)^2 \frac{1}{\sqrt{\pi}} \psi \sin\left(\frac{n+1}{2}\theta\right) d\theta$$

$$-\frac{n+1}{2\sqrt{\pi}} \cdot \psi(2\pi) \cdot \cos((n+1)\pi) \qquad + \quad C_n \cdot \left(\frac{n+1}{2}\right)^2$$

$$+ \frac{n+1}{2\sqrt{\pi}} \psi(0) \cdot \cos(0)$$

$\psi(2\pi) = \psi(0) \Rightarrow$ periodic BCs

$\cos(\pi) = \cos(0) = 1$

$\Rightarrow \; c_n \left(\frac{n+1}{2}\right)^2$

* Expand out derivatives of r

$$r\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = r\left[r\frac{d^2\psi}{dr^2} + \frac{d\psi}{dr}\right] = r^2\frac{d^2\psi}{dr^2} + r\frac{d\psi}{dr}$$

$$r^2\frac{d^2c_n}{dr^2} + r\frac{dc_n}{dr} = c_n\left(\frac{n+1}{2}\right)^2$$

$$\frac{d^2c_n}{dr^2} + \frac{1}{r}\frac{dc_n}{dr} - \frac{1}{r^2}c_n\left(\frac{n+1}{2}\right)^2 = 0$$

Equidimensional, form of $c_n(r) = r^m$. Plug in & get char Eq.

$$(m-1)m\,\cancel{r^{m-2}} + \cancel{\frac{1}{r}}\cdot m\,\cancel{r^{m-1}} - \cancel{\frac{1}{r^2}\,r^m}\cdot\left(\frac{n+1}{2}\right)^2 = 0$$

$$(m-1)m + m - \left(\frac{n+1}{2}\right)^2 = 0$$

$$m^2 - m + m - \left(\frac{n+1}{2}\right)^2 = 0$$

$$m^2 = \left(\frac{n+1}{2}\right)^2 \qquad m = \pm\frac{n+1}{2}$$

$$c_n(r) = Ar^{(n+1)/2} + Br^{-(n+1)/2}$$

$$c_n(R) = 0 \qquad c_n(\infty) = ur$$

Apply BC's in r

$$\cancel{\psi(r,\theta) = \sum_{n=1}^{\infty}}$$

$$0 = AR^{\frac{n+1}{2}} + BR^{-(n+1)/2}$$

$$C_n(\infty) = ur = \lim_{r\to\infty} A r^{(n+1)/2} + B \underset{\text{goes to } 0.}{r^{-(n+1)/2}} = A r^{(n+1)/2}$$

$$A r^{n+1/2} = ur \quad \Rightarrow A = u$$

$$\frac{n+1}{2} = 1 \Rightarrow n = 1$$

Fixed by BC!

$$C_n(r) = Ar + Br^{-1}$$

$$= ur + Br^{-1}$$

$$0 = uR + B/R \Rightarrow -uR^2 = B$$

$$C_n(r) = -uR^2 r^{-1} + ur$$

$$C_n(r) = uR\left(\frac{r}{R} + \frac{R}{r}\right)$$

if $n=1$, odd!

sin was correct choice.

$$\psi(r,\theta) = \sum_{n=1}^{\infty} C_n(r)\, D_n(\theta)$$

only need $n=1$

$$= C_1(r)\, D_1(\theta) = uR\left(\frac{r}{R} + R/r\right) \cdot \frac{1}{\sqrt{\pi}} \sin\theta$$

$$\frac{1}{\sqrt{\pi}} \sin\left(\left(\frac{n+1}{2}\right)\theta\right) = \frac{1}{\sqrt{\pi}} \sin\theta$$

$n=1$ $\frac{n+1}{2} = 1$

~~$\psi(R,\theta)$~~

* I needed to transform BC's.

$$\int_0^{2\pi} \frac{1}{\sqrt{\pi}} \sin\left(\frac{n+1}{2}\theta\right) \cdot ur\sin\theta \, d\theta$$

$$ur \int_0^{2\pi} \frac{1}{\sqrt{\pi}} \sin\left(\frac{n+1}{2}\theta\right) \sin\theta \, d\theta$$

let $n=1$

$$\frac{ur}{\sqrt{\pi}} \underbrace{\int_0^{2\pi} \sin^2\theta \, d\theta}_{\pi}$$

$$= ur\sqrt{\pi}$$

This cancels w/ $\frac{1}{\sqrt{\pi}}$ from above.

$$\boxed{\varphi(r,\theta) = uR\sin\theta\left(\frac{r}{R} + R/r\right)}$$

* Don't need an infinite sum because the BC for θ (eigenproblem) is <u>an eigenfunction!</u>