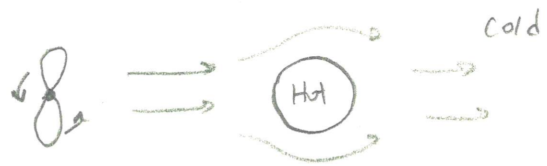
Lecture 22: Forced Convection - Internal Laminar Flows

I. Types of convective Heat and Mass Transfer

* Now that we have done heat/mass transport and fluid flow, we can put them together and do convective heat/mass transport problems

* There are two different types of convection problems: forced convection & free convection

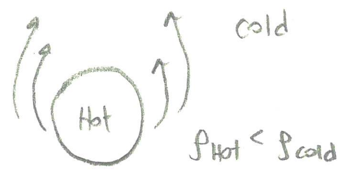
Forced



(applied pressure or shear)

Free (no fan)

(aka Natural)



(density driven flow)

* In forced convection, the velocity field is largely unaffected by the heat/mass transfer. We can solve for $\underline{v}$ first using a momentum balance. Then we can solve the heat/mass transfer problem.

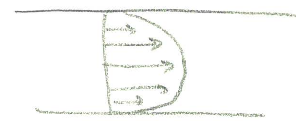
(In the end, we will get $Nu = Nu(Re, Pr)$)

* In free convection, the velocity field is created by the heat/mass transfer. So, we have to solve the energy (T) and momentum ($\underline{v}$) equations simultaneously

(In the end, we will get $Nu = Nu(Gr, Pr)$)

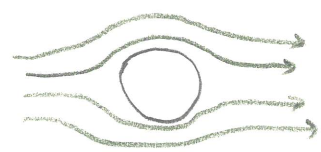
* Convection problems are also distinguished by the geometry of the flow.

Internal Flows
(Confined Flows)



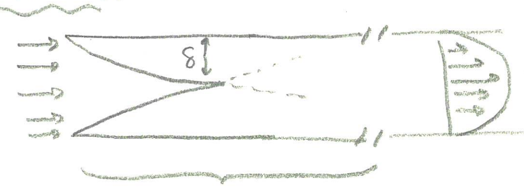
prototypical example: pipe flow

External Flows
(Unconfined Flows)



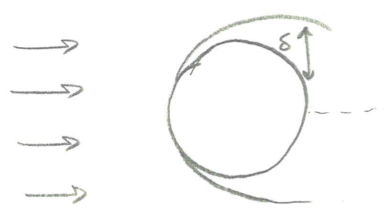
prototypical example: flow over a pipe.

* In <u>confined flows</u>, we have walls that enclose our flow. Consequently, at steady state, any boundary layers eventually grow to span the domain. Such a flow is considered <u>fully developed</u> and can often be <u>unidirectional</u>.



entrance region where BL grow.

* In <u>unconfined flows</u>, the walls are internal and we have some boundary condition at infinity. Consequently, at steady state we have finite boundary layers that we must consider (for flow at high Re). Such flows are almost always at least <u>bidirectional</u>.



* Finally, in both forced and free convection and for both confined and unconfined flows, we may have turbulence. The latter introduces serious complications to our analysis of these processes. For now, we will consider so-called "laminar" flows.

II. "Practical" Heat/Mass Transfer and Dimensional Analysis

* Let us consider forced-convection heat/mass transfer for a confined laminar flow.

Energy Equation $\quad \dfrac{\partial T}{\partial t} + \underline{v} \cdot \underline{\nabla} T = \alpha \nabla^2 T$

Species Mass Equation $\quad \dfrac{\partial c_i}{\partial t} + \underline{v} \cdot \underline{\nabla} c_i = D_i \nabla^2 c_i$

In the past, we let $\underline{v}$ be zero. No longer!

* Restrict ourselves to steady problems.

$$\frac{\partial T}{\partial t} = 0 \quad \Rightarrow \quad \underline{v} \cdot \underline{\nabla} T = \alpha \nabla^2 T$$

$$\frac{\partial c_i}{\partial t} = 0 \quad \Rightarrow \quad \underline{v} \cdot \underline{\nabla} c_i = D_i \nabla^2 c_i$$

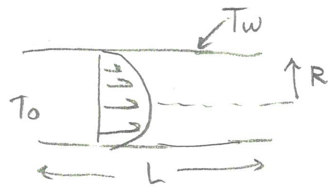
* How get $\underline{v}$? By solving the momentum equation (Navier Stokes)

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla} \underline{v}\right) = -\underline{\nabla} p + \mu \nabla^2 \underline{v}$$

$$\frac{\partial \underline{v}}{\partial t} = 0 \quad \Rightarrow \quad \underline{v} \cdot \underline{\nabla} \underline{v} = -\frac{1}{\rho} \underline{\nabla} p + \nu \nabla^2 \underline{v}$$

* Let us Non-dimensionalize this problem. We will focus on heat transfer for now. Note that the equations for H.T and M.T. (dilute) are the same!

Example:



T_0: inlet temp

$$\tilde{\nabla} = R\nabla \qquad \theta = \frac{T - T_w}{T_w - T_0} \qquad \tilde{\underline{v}} = \frac{\underline{v}}{u}$$

$$u\,\tilde{\underline{v}} \cdot \frac{\Delta T}{R}\tilde{\underline{\nabla}}\theta = \alpha \frac{\Delta T}{R^2}\tilde{\nabla}^2\theta$$

$$\frac{uR}{\alpha}\tilde{\underline{v}} \cdot \tilde{\underline{\nabla}}\theta = \tilde{\nabla}^2\theta \qquad Pe = \frac{uR}{\alpha} = \frac{u}{\alpha/R} \qquad \frac{\text{convective velocity}}{\text{conductive velocity}}$$

$$\boxed{Pe\ \tilde{\underline{v}} \cdot \tilde{\nabla}\theta = \tilde{\nabla}^2\theta}$$

Also recall that: (when $\tilde{p} = P/\rho u^2$)

$$\frac{u^2}{R}\tilde{\underline{v}} \cdot \tilde{\underline{\nabla}}\tilde{\underline{v}} = -\frac{1}{\rho}\frac{\rho u^2}{R}\tilde{\nabla}\tilde{p} + \frac{\mu}{\rho}\frac{u}{R^2}\tilde{\nabla}^2\tilde{\underline{v}}$$

$$\tilde{\underline{v}} \cdot \tilde{\underline{\nabla}}\tilde{\underline{v}} = -\tilde{\underline{\nabla}}\tilde{p} + \frac{\mu}{\rho R u}\tilde{\nabla}^2\tilde{\underline{v}}$$

$$\boxed{\tilde{\underline{v}} \cdot \tilde{\underline{\nabla}}\tilde{\underline{v}} = -\tilde{\underline{\nabla}}\tilde{p} + \frac{1}{Re}\tilde{\nabla}^2\tilde{\underline{v}}} \qquad Re = \frac{\rho u R}{\mu}$$

* From this we conclude that

$$\tilde{\underline{v}} = \tilde{v}(\tilde{r}, Re, \text{geometric ratios})$$ ratios of length scales. Eg. $\frac{R}{L}$

$$\theta = \theta(\tilde{r}, Re, Pe, \text{geometric ratios})$$

- Sometimes the latter is written as

$$\theta = \theta(\tilde{r}, Re, Pr, \text{geometric ratios}) \quad (HT)$$

$$\theta = \theta(\tilde{r}, Re, Sc, \text{geometric ratios}) \quad (MT)$$

$$Pr = \frac{\nu}{\alpha} \qquad Sc = \frac{\nu}{D_i} \qquad \text{(ratios of "diffusivities")}$$

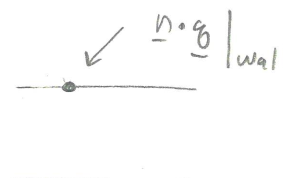
$$(HT) \quad Pe = Re\,Pr = \frac{\rho U R}{\mu} \cdot \frac{\nu}{\alpha} = \frac{UR}{\alpha} \checkmark$$

$$(MT) \quad Pe = Re\,Sc = \frac{\rho U R}{\mu} \cdot \frac{\nu}{D_i} = \frac{UR}{D_i} \checkmark$$

* In "practical"/engineering contexts, we want to know the total heat transfer (mass transfer).

$$Q = \int \underline{n} \cdot \underline{q}\Big|_{wall} dA \quad \text{or} \quad \dot{m}_i = \int \underline{n} \cdot \underline{N}_i\Big|_{wall} dA$$

* Focusing on H.T. (just to be succinct), we want to know $\underline{q}$, the heat flux.



$$\underline{n} \cdot \underline{q}\Big|_{wall} = -k(\underline{n} \cdot \nabla T)\Big|_{wall} = -k \frac{\partial T}{\partial n}\Big|_{wall}$$

* Conventionally, engineers prefer to write heat fluxes using a heat transfer coefficient

$$q'' = h(T_b - T_w) \qquad (q'' \text{ in Incopera})$$

T_w ← wall temperature

T_b ← bulk or mean temperature
- velocity averaged
- "flow rate" averaged

$$T_b = \frac{\int v_z T \, dA}{\int v_z \, dA}$$

- think of this as a definition of h. A way to conveniently express heat fluxes.

$$h = \frac{-k \, \partial T/\partial n \,|_{wall}}{T_b - T_w}$$

* we would prefer to write this in a dimensionless form.

$$h = \frac{-k \cdot \frac{\cancel{\Delta T}}{R} \cdot \frac{\partial \theta}{\partial \tilde{n}}\Big|_w}{\cancel{\Delta T}\,(\theta_b - \theta_w)} \Rightarrow \frac{hR}{k} = -\frac{\frac{\partial \theta}{\partial \tilde{n}}\Big|_w}{\theta_b - \theta_w}$$

$Nu = \frac{hR}{k}$ Nusselt number. (dimensionless HT coefficient)

$$\boxed{Nu = \frac{-\partial\theta/\partial\tilde{n}\,|_w}{\theta_b - \theta_w}}$$

* If we can get Nu for a given problem, then we can use it for engineering calculations.

* From our dimensional analysis, we conclude that

$Nu = Nu(\tilde{r}, Re, Pr, \text{geometric ratios})$ just like θ

- if we average over the domain:

$$\boxed{\overline{Nu} = \overline{Nu}(Re, Pr, \text{geometric ratios})}$$

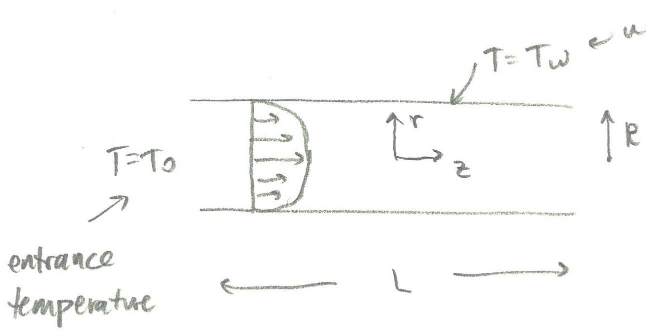
* This follows similarly for mass transfer

$Sh = \frac{k_i R}{D_i}$, k_i: mass transfer coefficient for species i

$Sh = Sh(\tilde{r}, Re, Sc_i, \text{geometric ratios})$

III. The Graetz Problem (setup)

* The Graetz problem is a classical example of heat transfer in confined flow. It assumes we have laminar flow in a tube.



cylindrical coordinates

$\frac{\partial T}{\partial \theta} = 0$, symmetric in θ

$$v_z = 2u\left(1 - \frac{r^2}{R^2}\right)$$

u: mean velocity
$(v_r = v_\theta = 0)$

* we want to solve for $T(r,z)$ and then use this to get Nu.

* Energy equation: $\underline{v} \cdot \underline{\nabla} T = \alpha \nabla^2 T$

$$\underbrace{v_r \frac{\partial T}{\partial r}}_{v_r = 0} + \underbrace{\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}}_{v_\theta = 0} + v_z \frac{\partial T}{\partial z} = \alpha\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}}_{\frac{\partial T}{\partial \theta} = 0} + \frac{\partial^2 T}{\partial z^2}\right]$$

$$v_z \frac{\partial T}{\partial z} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \alpha\frac{\partial^2 T}{\partial z^2}$$

$$\boxed{2u\left(1 - \frac{r^2}{R^2}\right)\frac{\partial T}{\partial z} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \alpha\frac{\partial^2 T}{\partial z^2}}$$ ← PDE for $T(r,z)$

Boundary Conditions

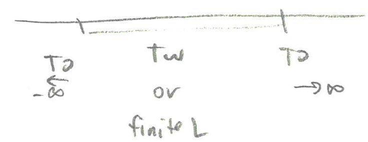
$\boxed{T(r,0) = T_0}$ $\boxed{T(R,z) = T_w}$

* other z B.C. not important.

$\boxed{\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0}$ (symmetry)

* Aside: This is the "extended" Graetz problem with axial conduction term. I see people solve it where:

$\alpha \frac{\partial^2 T}{\partial z^2}$



$T(r,0) = T_o$ and $T(r,L) = T_o$

or $T(r,-\infty) = T_o$ and $T(r,\infty) = T_o$

or possibly $T(r,0) = T_o$ and $\left.\frac{dT}{dz}\right|_{z=\infty} = 0$

These B.C's will likely all give the same answer, as long as L is big enough. $T = T(z)$ should be flat in the middle of the tube (axially)

* Let's non-dimensionalize:

$$\eta = \frac{r}{R}, \quad \theta = \frac{T - T_W}{T_0 - T_W}, \quad \text{figure out } z \text{ in a minute (O.O.M. analysis)}$$

$\Delta T = T_0 - T_W$

* Good when $T_0 > T_W$. Can switch to $\theta = \frac{T - T_0}{T_W - T_0}$ when $T_W > T_0$

$\zeta = z/\ell \leftarrow$ what ℓ?

$$2u(1-\eta^2) \cdot \frac{\Delta T}{\ell} \frac{\partial \theta}{\partial \zeta} = \frac{\alpha \Delta T}{R^2} \frac{1}{\eta} \frac{\partial}{\partial \eta}\left(\eta \frac{\partial \theta}{\partial \eta}\right) + \frac{\alpha \Delta T}{\ell^2} \frac{\partial^2 \theta}{\partial \zeta^2}$$

$$(1-\eta^2) \frac{\partial \theta}{\partial \zeta} = \frac{\alpha \ell}{2uR^2} \frac{1}{\eta} \frac{\partial}{\partial \eta}\left(\eta \frac{\partial \theta}{\partial \eta}\right) + \frac{\alpha}{2u\ell} \frac{\partial^2 \theta}{\partial \zeta^2}$$

$$\frac{\alpha}{2uR} \cdot \frac{R}{\ell}$$

let $Pe = \frac{2uR}{\alpha}$

$$(1-\eta^2) \frac{\partial \theta}{\partial \zeta} = \frac{1}{Pe} \frac{\ell}{R} \frac{1}{\eta} \frac{\partial}{\partial \eta}\left(\eta \frac{\partial \theta}{\partial \eta}\right) + \frac{1}{Pe} \frac{R}{\ell} \frac{\partial^2 \theta}{\partial \zeta^2}$$

2 groups to deal with

• Some choices: 1) $\ell = L \gg R$

$$\underbrace{\frac{1}{Pe}\frac{L}{R}}_{?} \quad \& \quad \underbrace{\frac{1}{Pe}\frac{R}{L}}_{\text{really small}}$$

• we need to know how $L \leftrightarrow Pe$ or else can't tell the scale.

usually $Pe \gg 1$

Flow in a pipe is pretty fast.

2) $\ell = R$

$$\frac{1}{Pe}\cdot\frac{R}{R} \quad \& \quad \frac{1}{Pe}\cdot\frac{R}{R}$$

• unrealistically short pipe

"what measuring stick should we choose?"

• Like a semi-infinite problem for z.

what scale for z?

3) $\ell = Pe\,R$ (will have $\ell \gg R$)

$$\frac{1}{Pe}\cdot\frac{Pe\,R}{R} = 1 \qquad \frac{1}{Pe}\cdot\frac{R}{Pe\,R} = \frac{1}{Pe^2}$$

This gives us the scale for z, where radial conduction and convection are balanced. Just as much heat is being conducted in as flow is carrying away.

A special "L" where the terms balance. Need $\frac{\partial T}{\partial z}$ to be small enough for convection to match radial conduction.

• why do I have a D.O.F. to pick z?

Because at high Pe, I don't have another B.C. because I lose the axial diffusion term. Just like semi-infinite diffusion

+ using $\zeta = z/R\,Pe$ gives

$$(1-\eta^2)\frac{\partial\theta}{\partial\zeta} = \frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\theta}{\partial\eta}\right) + \frac{1}{Pe^2}\frac{\partial^2\theta}{\partial\zeta^2}$$

+ we typically care about the case when $Pe \gg 1$.

In this case, axial (down the pipe) diffusion is not important.

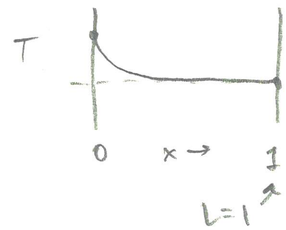
$$\boxed{\begin{gathered}(1-\eta^2)\frac{\partial\theta}{\partial\zeta} = \frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\theta}{\partial\eta}\right)\\ \theta(\eta,\zeta=0) = 1, \quad \theta(\eta=1,\zeta) = 0, \quad \frac{\partial\theta}{\partial\eta}(\eta=0,\zeta) = 0\end{gathered}}$$

Dimensionless Graetz problem.

IV. Solution to the Graetz Problem

Skip straight to the FFT series solution. p. 16

* There are two useful solutions to the Graetz problem. A useful analogy here is the transient diffusion problem:



$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad T(x,0)=0 \quad T(0,t)=1 \quad T(1,t)=0$$

$$T(x,t) = \underbrace{1-x}_{T_{ss}} - 2\sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n\pi} e^{-n^2\pi^2 t}$$

- at early times $(t \ll L/D^2)$

the similarity solution : $T = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-s^2} ds \quad , \eta = \frac{x}{\sqrt{4\alpha t}}$

gives an accurate solution.

(Note: at early times, the similarity solution is preferred, because the series solution converges slowly for small t.)

* Similarly for the Graetz problem, we have a series solution & a similarity solution. Note the analogy between the ζ variable and time in the transient diffusion problem. ↳ (this is space in z-axis, with diffusion now being radial for Graetz).

  - So the small-ζ solution for the Graetz problem is an "entrance region" rather than "early time."

A. Entrance Region Solution for Graetz problem (similarity)

This analysis uses: $\theta = \dfrac{T-T_o}{T_w-T_o}$

$$(1-\eta^2)\frac{\partial\theta}{\partial\zeta} = \frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\theta}{\partial\eta}\right)$$

let $x = 1-\eta$ so x is small when $\zeta \ll 1 \Rightarrow x \ll 1$

$$\left[1-(1-x)^2\right]\frac{\partial\theta}{\partial\zeta} = \frac{1}{(1-x)}\frac{\partial}{\partial(1-x)}\left[(1-x)\frac{\partial\theta}{\partial(1-x)}\right]$$

$$\left[\cancel{1}-\cancel{1}+2x-x^2\right]\frac{\partial\theta}{\partial\zeta} = \underbrace{\frac{1}{1-x}\frac{\partial}{\partial x}\left[(1-x)\frac{\partial\theta}{\partial x}\right]}$$

(linear velocity profile) → $x^2 \ll x$ ← (*) → $1-x \approx 1$ ← (essentially low curvature)

$$\boxed{2x\frac{\partial\theta}{\partial\zeta} = \frac{\partial^2\theta}{\partial x^2}}$$

(*) Is called the Lévêque approximation.

$\boxed{\theta(x,0)=0}$, $\boxed{\theta(0,\zeta)=1}$, $\boxed{\theta(\infty,\zeta)=0}$

↑ replaces symmetry.
Says $T=T_o$ in center.

* Now, suppose that

$$\theta = \theta(s) \quad \text{where} \quad s = \frac{x}{g(\zeta)} \qquad \zeta = \frac{z}{R\,Pe} \qquad x = 1-\frac{r}{R}$$

* Convert the PDE to similarity variables.

$$\frac{\partial\theta}{\partial\zeta} = \frac{\partial s}{\partial\zeta}\frac{\partial\theta}{\partial s} \qquad \frac{\partial s}{\partial\zeta} = \frac{\partial}{\partial\zeta}\left(\frac{x}{g}\right) =$$

$$= x\left(-g^{-2}\right)g'$$

$$= -\frac{xg'}{g^2}\frac{\partial\theta}{\partial s} \qquad = -\frac{xg'}{g^2}$$

$$\frac{\partial\theta}{\partial x} = \frac{\partial\theta}{\partial s}\frac{\partial s}{\partial x} = \frac{\partial\theta}{\partial s}\cdot\frac{\partial}{\partial x}\left(\frac{x}{g}\right) = \frac{1}{g}\frac{\partial\theta}{\partial s}$$

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\theta}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{g}\frac{\partial\theta}{\partial s}\right) = \frac{1}{g}\frac{\partial}{\partial x}\left(\frac{\partial\theta}{\partial s}\right)$$

$$= \frac{1}{g}\frac{\partial s}{\partial x}\frac{\partial}{\partial s}\left(\frac{\partial\theta}{\partial s}\right) = \frac{1}{g}\cdot\frac{1}{g}\frac{\partial^2\theta}{\partial s^2} = \frac{1}{g^2}\frac{\partial^2\theta}{\partial s^2}$$

* Put it together

$$2x\frac{(-xg')}{g^2}\frac{\partial\theta}{\partial s} = \frac{1}{g^2}\frac{\partial^2\theta}{\partial s^2}$$

$$-\frac{2x^2}{g^2}g'\frac{\partial\theta}{\partial s} = \frac{1}{g^2}\frac{\partial^2\theta}{\partial s^2} \qquad s = \frac{x}{g}$$

$$-2s^2g'\frac{\partial\theta}{\partial s} = \frac{1}{g^2}\frac{\partial^2\theta}{\partial s^2}$$

$$\boxed{\frac{\partial^2\theta}{\partial s^2} + 2s^2g^2g'\frac{\partial\theta}{\partial s} = 0}$$

* Solve for g:

$$g^2\frac{dg}{d\xi} = c \qquad g^2\frac{dg}{d\xi} = \frac{d}{d\xi}(g^3)\cdot\frac{1}{3}$$

$$\int\frac{1}{3}d(g^3) = \int c\,d\xi \Rightarrow \frac{1}{3}g^3 = c\xi + c_1$$

let $g(0) = 0 \Rightarrow c_1 = 0$

$$g^3 = 3\xi\cdot c \qquad g = (3\xi\cdot c)^{1/3}$$

Deen picks $\underline{c = 3/2} \Rightarrow \boxed{g = \left(\frac{9}{2}\xi\right)^{1/3}}$

$g^2g' = \frac{3}{2}$
plug into above

$$\boxed{s = \frac{x}{g} = \left(\frac{2}{9}\right)^{1/3}\frac{x}{\xi^{1/3}}}$$

* now fix BCs for similarity problem

$$\theta(x,0)=0 \Rightarrow s=\left(\tfrac{2}{9}\right)^{1/3}\cdot\frac{x}{0^{1/3}}=\infty$$

$$\boxed{\theta(\infty)=0}$$

$$\theta(0,\xi)=1 \Rightarrow s=\left(\tfrac{2}{9}\right)^{1/3}\cdot\frac{0}{\xi^{1/3}}=0$$

$$\boxed{\theta(0)=1}$$

$$\theta(\infty,\xi)=0 \Rightarrow s=\left(\tfrac{2}{9}\right)^{1/3}\cdot\frac{\infty}{\xi^{1/3}}=\infty$$

* Now solve similarity ODE

$$\frac{d^2\theta}{ds^2}+3s^2\frac{d\theta}{ds}=0 \qquad \theta(0)=1\ ,\ \theta(\infty)=0$$

let $u=\dfrac{d\theta}{ds}$ $\quad \dfrac{du}{ds}=\dfrac{d^2\theta}{ds^2}$

$$\frac{du}{ds}+3s^2u=0 \Rightarrow \frac{du}{ds}=-3s^2u$$

$$\Rightarrow \frac{1}{u}\frac{du}{ds}=-3s^2 \Rightarrow \ln u=-s^3+c_1$$

$$u=\exp(-s^3)\,c_1'$$

$$\frac{d\theta}{ds}=c_1'\exp(-s^3)$$

$$\theta=\int_0^s c_1'\exp(-t^3)\,dt+c_2$$

- Apply B.C's: $\theta(0)=1$
  $\theta(\infty)=0$

$$\theta(0) = \int_0^0 c_1' \exp(-t^3) + c_2 = 1 \quad \Rightarrow c_2 = 1$$

$$\theta(\infty) = \int_0^\infty c_1' \exp(-t^3) + 1 = 0$$

$$c_1' = \frac{-1}{\int_0^\infty \exp(-t^3)}$$

$$\theta(s) = \frac{-\int_0^s \exp(-t^3)\,dt}{\int_0^\infty \exp(-t^3)\,dt} + 1$$

can solve via a trick with gamma functions and u-substitution

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\,dx \quad \text{(gamma func)}$$

$$I = \int_0^\infty e^{-t^3}\,dt$$

let $x = t^3 \Rightarrow t = x^{1/3}$

$$dt = \tfrac{1}{3} x^{-2/3}\,dx$$

$$= \int_0^\infty e^{-x} \cdot \tfrac{1}{3} x^{-2/3}\,dx$$

$$= \frac{1}{3}\int_0^\infty \underset{\uparrow \atop x^{1/3-1}}{x^{-2/3}} e^{-x}\,dx = \frac{1}{3}\Gamma(1/3)$$

$\Rightarrow z = 1/3$

So,
$$\theta(s) = -\frac{\int_0^s \exp(-t^3)\,dt}{\frac{1}{3}\Gamma(1/3)} + 1 = \frac{-\int_0^s e^{-t^3}dt + \int_0^\infty e^{-t^3}dt}{\frac{1}{3}\Gamma(1/3)}$$

$$\boxed{\theta(s) = \frac{3}{\Gamma(1/3)} \int_s^\infty \exp(-t^3)\,dt}, \quad \boxed{s = \left(\frac{2}{9}\right)^{1/3} \frac{x}{\zeta^{1/3}}}$$

* Now, to find the Nusselt number:

$$Nu = \frac{2hR}{k} = \frac{-2\,\partial\theta/\partial x\big|_{x=0}}{\theta_w - \theta_b} = -2\frac{\partial\theta}{\partial x}\bigg|_{x=0}$$

for tube diameter

$\theta(0)=1$ $\qquad$ $\theta(\infty)=0$

$$\frac{d\theta}{dx} = \frac{d\theta}{ds}\frac{ds}{dx} = \frac{1}{g}\frac{d\theta}{ds} \qquad s = x/g$$

(from above) $\quad \dfrac{d\theta}{ds} = c_1' \exp(-s^3) \qquad c_1' = \dfrac{-1}{\int_0^\infty \exp(-t^3)} = \dfrac{-3}{\Gamma(1/3)}$

$$\frac{d\theta}{ds} = \frac{-3}{\Gamma(1/3)}\exp(-s^3)$$

$$\frac{d\theta}{ds}\bigg|_{x=0} = \frac{d\theta}{ds}\bigg|_{s=0} = \frac{-3}{\Gamma(1/3)}$$

$$Nu = -2\cdot\frac{1}{g}\cdot\frac{(-3)}{\Gamma(1/3)} = \frac{6}{\Gamma(1/3)\,g(\zeta)}$$

$$\boxed{Nu = \frac{6}{\Gamma(1/3)}\frac{1}{g(\zeta)}} \qquad g(\zeta) = \left(\frac{9}{2}\zeta\right)^{1/3} \qquad \zeta = \frac{z}{R\,Pe}$$

$$\boxed{Nu = \underbrace{\frac{6}{\Gamma(1/3)}\left(\frac{2}{9}\right)^{1/3}}_{1.3566}\cdot\left(\frac{R}{z}\right)^{1/3}(Pe)^{1/3}}$$

*First term in asymptotic series due to Newman (1967, 1969):

$$Nu = 1.3565975\,\zeta^{-1/3} - 1.2 - 0.296919\,\zeta^{1/3} + O(\zeta^{2/3})$$

B. Fully Developed Solution for Graetz Problem (FFT)

$$\frac{\partial \theta}{\partial \zeta} = \frac{1}{\eta(1-\eta^2)} \frac{\partial}{\partial \eta}\left(\eta \frac{\partial \theta}{\partial \eta}\right)$$

let $\theta = \dfrac{T - T_w}{T_0 - T_w}$

it swaps the BCs compared to similarity soln.

These two "swap" →

$\theta(\eta, \zeta=0) = 1 \quad \longleftarrow \quad T(r,0) = T_0$

$\theta(\eta=1, \zeta) = 0 \quad \longleftarrow \quad T(R,z) = T_w$

$\frac{\partial \theta}{\partial \eta}(\eta=0, \zeta) = 0 \quad \longleftarrow \quad \frac{\partial T}{\partial r}(0,z) = 0$

Eigenfunction:

$$\frac{1}{\eta(1-\eta^2)} \frac{d}{d\eta}\left(\eta \frac{d\Phi}{d\eta}\right) = -\lambda^2 \Phi$$

is Sturm-Liouville ⇒ Cylindrical Graetz functions

$$\Phi_n(\eta) = a_n G(\lambda_n \eta)$$

↖ normalization constant.

$$w(\eta) = (1-\eta^2)\eta, \quad p(\eta) = \eta, \quad q(\eta) = 0$$

FFT:

$$\hat{\theta}_n = \int_0^1 \theta \cdot \Phi_n (1-\eta^2)\eta \, d\eta$$

IFFT:

$$\theta(\eta, \zeta) = \sum_{n=1}^{\infty} \hat{\theta}_n(\zeta) \Phi_n(\eta)$$

- Apply to PDE gives:

LHS: $\displaystyle\int_0^1 \frac{\partial \theta}{\partial \zeta} \Phi_n (1-\eta^2)\eta \, d\eta$

$$= \frac{\partial}{\partial \zeta} \int_0^1 \theta \Phi_n (1-\eta^2)\eta \, d\eta = \frac{\partial \hat{\theta}_n}{\partial \zeta}$$

RHS :
$$\int_0^1 \underbrace{\frac{1}{\eta(1-\eta^2)}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\theta}{\partial\eta}\right)}_{\text{By the eigenvalue problem this equals } -\lambda_n^2\theta} \cdot \Phi_n(1-\eta^2)\,\eta\,d\eta$$

$$-\lambda^2\int_0^1 \theta\,\Phi_n(1-\eta^2)\,\eta\,d\eta = -\lambda_n^2\hat{\theta}_n$$

so:
$$\frac{\partial\hat{\theta}_n}{\partial\xi} = -\lambda_n^2\hat{\theta}_n \Rightarrow \hat{\theta}_n = \exp(-\lambda_n^2\xi)\cdot b_n$$

$$b_n = \int_0^1 \underbrace{\theta(\eta,\xi=0)}_{1}\,\Phi_n(1-\eta^2)\,\eta\,d\eta \quad \text{(FFT of initial Condition)}$$

$$b_n = \int_0^1 \Phi_n(1-\eta^2)\,\eta\,d\eta$$

* Put it all together:

$$\theta(\eta,\xi) = \sum_{n=1}^{\infty} b_n \exp(-\lambda_n^2\xi)\cdot a_n\,G(\lambda_n\eta)$$

$$\boxed{\theta(\eta\,\xi) = \sum_{n=1}^{\infty} c_n \exp(-\lambda_n^2\xi)\,G(\lambda_n\eta)}\quad, c_n = a_n b_n$$

* The trick to this one is getting the Graetz functions and the eigenvalues.

* Apparently, Graetz functions are more easily found via "confluent hypergeometric functions"

$$G(\lambda\eta) = e^{-\lambda\eta^2/2} M\left(\tfrac{1}{2}-\tfrac{\lambda}{4}, 1, \lambda\eta^2\right)$$

with eigenvalues given by zeros:

$$M\left(\tfrac{1}{2}-\tfrac{\lambda}{4}, 1, \lambda\right) = 0$$

* Now to get Nu:

$$Nu(\zeta) = \frac{2hR}{k} = -\frac{2(\partial\theta/\partial\eta)\big|_{\eta=1}}{\theta_b(\zeta) - \theta_w} \quad \leftarrow \theta_w = \theta(\eta=1) = 0$$

$$\frac{\partial\theta}{\partial\eta} = \frac{\partial}{\partial\eta}\sum_{n=1}^{\infty} c_n e^{-\lambda_n^2\zeta} G(\lambda_n\eta) \qquad \text{(go term by term)}$$

$$= \sum_{n=1}^{\infty} c_n e^{-\lambda_n^2\zeta} \frac{\partial G}{\partial\eta}(\lambda_n\eta) \qquad \leftarrow \text{take care of the } \lambda_n \text{ when do } \frac{\partial G}{\partial\eta} \text{ (see supp notes)}$$

$\eta = 1$

$$\frac{\partial\theta}{\partial\eta}\bigg|_{\eta=1} = \sum_{n=1}^{\infty} c_n e^{-\lambda_n^2\zeta} \frac{\partial G}{\partial\eta}(\lambda_n)$$

• Need to get θ_b:

$$T_b = \frac{\int T\, v_z\, dA}{\int v_z\, dA}$$

$T = \Delta T\,\theta$

$v_z = 2u(1 - r^2/R^2)$

$dA = 2\pi r\, dr$

$r = R\eta$

$$T_b = \frac{\int \Delta T\, \theta \cdot 2u(1-\eta^2) \cdot 2\pi r\, dr}{\int 2u(1-\eta^2) \cdot 2\pi r\, dr}$$

$$\frac{T_b}{\Delta T} = \frac{\int_0^1 \theta(1-\eta^2)\,\eta\, d\eta}{\int_0^1 (1-\eta^2)\,\eta\, d\eta}$$

→ do this one: $\int_0^1 (\eta-\eta^3)\,d\eta = \frac{\eta^2}{2} - \frac{\eta^4}{4}\Big|_0^1 = \frac{1}{2} - \frac{1}{4} = 1/4$

$$\theta_b = 4\int_0^1 \theta(1-\eta^2)\,\eta\, d\eta$$

- Using the Graetz PDE:

$$(1-\eta^2)\frac{d\theta}{d\zeta} = \frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\theta}{\partial\eta}\right)$$

↓

$$\frac{\partial}{\partial\zeta}\underbrace{\int_0^1 \theta\cdot\eta(1-\eta^2)\,d\eta}_{\theta_b/4} = \int_{\eta=0}^{\eta=1} d\left(\eta\frac{\partial\theta}{\partial\eta}\right)$$

$$\frac{1}{4}\frac{\partial\theta_b}{\partial\zeta} = \left(\eta\frac{\partial\theta}{\partial\eta}\right)\Big|_{\eta=1} - \underbrace{\left(\eta\frac{\partial\theta}{\partial\eta}\right)\Big|_{\eta=0}}_{\text{this term is zero.}}$$

↑ this term is $\eta=1,\ \frac{\partial\theta}{\partial\eta}\Big|_{\eta=1}$

$$\frac{\partial\theta_b}{\partial\zeta} = 4\frac{\partial\theta}{\partial\eta}\Big|_{\eta=1}$$

integral w.r.t. ζ

so, $\theta_b = -4\int_\zeta^\infty \frac{\partial\theta}{\partial\eta}(1,t)\,dt$ ($T_b \to T_w$) (B/C $\theta_b \to 0$ as $\zeta \to \infty$)

Integrate, term by term → $\theta_b = -4\int_\zeta^\infty \sum_{n=1}^{\infty} C_n e^{-\lambda_n^2 t}\frac{\partial G}{\partial\eta}(\lambda_n)\,dt$

$$= -4\sum_{n=1}^{\infty} C_n \cdot \frac{e^{-\lambda_n^2 \zeta}}{\lambda_n^2}\frac{\partial G}{\partial\eta}(\lambda_n)$$

negative sign stays: $\left[\frac{e^{-\lambda^2\cdot\infty}}{-\lambda^2} - \frac{e^{-\lambda^2\zeta}}{-\lambda^2}\right]$

$$\theta_b = -4 \sum_{n=1}^{\infty} c_n \frac{e^{-\lambda_n^2 \xi}}{\lambda_n^2} \frac{\partial G}{\partial \eta}(\lambda_n)$$

* Putting it all together

$$Nu = \frac{-2 \sum_{n=1}^{\infty} c_n e^{-\lambda_n^2 \xi} \frac{\partial G}{\partial \eta}(\lambda_n)}{-4 \sum_{n=1}^{\infty} c_n \frac{e^{-\lambda_n^2 \xi}}{\lambda_n^2} \frac{\partial G}{\partial \eta}(\lambda_n)}$$

$$\boxed{Nu = \frac{1}{2} \frac{\sum_{n=1}^{\infty} c_n e^{-\lambda_n^2 \xi} \frac{\partial G}{\partial \eta}(\lambda_n)}{\sum_{n=1}^{\infty} c_n \frac{e^{-\lambda_n^2 \xi}}{\lambda_n^2} \frac{\partial G}{\partial \eta}(\lambda_n)}}$$

keeping only $n=1$ (good when $\xi \to \infty$, λ_1 dominates because it is smallest.

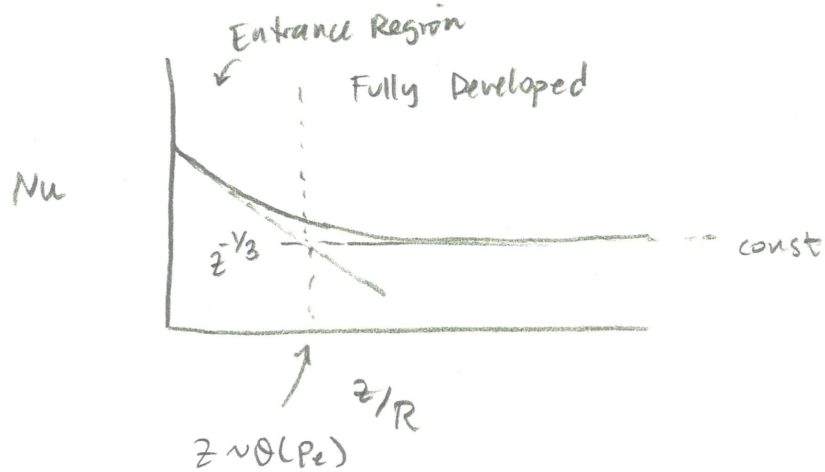
$$Nu\,(\xi \to \infty) = \frac{1}{2} \frac{\cancel{c_1} \cancel{e^{-\lambda_1^2 \xi}} \cdot \cancel{\frac{\partial G}{\partial \eta}(\lambda_1)}}{\cancel{c_1} \frac{\cancel{e^{-\lambda_1^2 \xi}}}{\lambda_1^2} \cdot \cancel{\frac{\partial G}{\partial \eta}(\lambda_1)}} = \frac{1}{2}\lambda_1^2$$

$$\boxed{Nu(\infty) = \frac{1}{2}\lambda_1^2}$$

From book $\lambda_1 = 2.7044$ so $\boxed{Nu(\infty) = 3.657}$

A constant!

* Conclusion. what does the solution look like?



* Entrance region: radial diffusion is growing

* Fully developed: radial diffusion meets wall and "maxes out"